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Cosmology of Brane Universes and Brane Gases

THÈSE

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Abstract

The standard big bang model gives a fairly good description of the cosmological evolution of our universe from shortly after the big bang to the present. The existence of an initial singularity, however, might be viewed as unsatisfactory in a comprehensive model of the universe. Moreover, if this singularity indeed exists, we are lacking initial conditions which tell us in what state the universe emerged from the big bang.

The advent of string theory as a promising candidate for a theory of quantum gravity opened up new possibilities to understand our universe. The hope is that string theory can resolve the initial singularity problem and, in addition, provide initial conditions.

String theory makes a number of predictions such as extra-dimensions, the existence of p-branes (fundamental objects with p spatial dimensions) as well as several new particles. Consequently, over the past few years, a new field of research emerged, which investigates how these predictions manifest themselves in a cosmological context. In particular, the idea that our universe is a 3-brane embedded in a higher-dimensional space received a lot of attention.

In this thesis we investigate the dynamical and perturbative behavior of string theory inspired cosmological models. After an introduction to extra-dimensions and p-branes, we identify our universe with a 3-brane embedded in a 5-dimensional bulk space-time. We then study the cosmology and the evolution of perturbations due to the motion of this brane through the higher-dimensional space-time. Secondly, we point out the dynamical instabilities of the Randall-Sundrum model. And thirdly, vector perturbations on the brane induced by perturbations in the bulk are calculated, and the resulting CMB power spectrum is estimated. In all cases, dynamical instabilities are encountered, which suggests that existing attempts to realize cosmology on branes are at least questionable.

In the last part of this thesis, we study string and brane gas cosmology. In this scenario, the role of strings and branes is to drive the background dynamics. A string theory specific symmetry between large and small scales (T-duality) is used to avoid the initial big bang singularity. We show that the evolution of an initially small and compact nine-dimensional space-time leads to three large dimensions, which then become our visible universe. Brane gas cosmology seems to us very promising to bring together string theory and cosmology.

This work is only part of a tremendously growing flood of literature, but we hope that it is nevertheless a piece of mosaic in the work of art which is science. We hope that the interplay between string theory and cosmology enables us to advance to new spheres and dimensions (see Fig. 1).

Figure 1:

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Examinateurs

Le jury de cette thèse se compose de

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Meinen Eltern gewidmet

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Résumé

Introduction

La théorie des cordes est une théorie fondamentale, qui unifie la gravitation et les interactions de jauge d'une manière consistante et renormalisable. Étant une théorie de la gravité quantique, on pense, qu'elle a joué un rôle important tôt dans l'histoire de l'univers, et qu'elle est nécessaire à la compréhension de celui-ci. D'autre part, due à la recolte d'une grande quantité de données astrophysiques, la cosmologie est devenue une science de précision. On espère pouvoir tester pour la première fois les prédictions de la théorie des cordes dans les processus cosmologiques.

Pendant les années passées l'interaction entre la théorie des cordes et la cosmologie est devenue un domaine de recherche important, qui a apporté de nouvelles connaissances aux deux sujets. L'objectif de cette thèse est d'étudier, comment se manifestent certaines prédictions de la théorie des cordes dans un cadre cosmologique. Cette thèse comporte quatre articles, qui ont été elaborés en collaboration avec d'autres chercheurs, ainsi que des chaptires d'introduction et de revue.

Le modèle standard de la cosmologie

Le modèle standard de la cosmologie s'appuie sur trois piliers: l'isotropie de l'expansion cosmique, l'isotropie du fond de rayonnement diffus ainsi que la synthèse des éléments légers. La géometrie d'un univers isotrope autour de chaque point (et donc homogène) est donnée par la métrique de Friedmann-Lemaître,

$$\mathrm{d}s_4^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}\tau^2 + a^2(\tau)\left[\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{1}$$

où τ est le temps cosmique, $a(\tau)$ le facteur d'échelle, et $\mathcal{K}$ la courbure des surfaces à τ constant. La dynamique du champ gravitationnel $g_{\mu\nu}$ est détérminée par les équations d'Einstein,

$$\mathcal{G}_{\mu\nu} + \Lambda_4 g_{\mu\nu} = 8\pi G_4 T_{\mu\nu}, \tag{2}$$

où le tenseur d'Einstein $\mathcal{G}_{\mu\nu}$ décrit la géometrie courbée de l'espace-temps, et le tenseur d'énergie-impulsion $T_{\mu\nu}$ son contenu matériel. En supposant que celui-ci est un fluide parfait, on a $T^\mu{}_\nu = \mathrm{diag}(-\rho, P, P, P)$. La quantité Λ_4 est la constante cosmologique quadridimensionnelle. Avec la métrique (1) les équations d'Einstein donnent

$$H^2 + \frac{\mathcal{K}}{a^2} = \frac{8\pi G_4}{3}\rho + \frac{\Lambda_4}{3}, \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_4}{3}(\rho + 3P) + \frac{\Lambda_4}{3}. \tag{4}$$

Le point désigne une dérivée par rapport à τ, et $H \equiv \dot{a}/a$ est le paramètre de Hubble. De l'identité de Bianchi, $\nabla_\nu \mathcal{G}^{\mu\nu} = 0$, et ainsi $\nabla_\nu T^{\mu\nu} = 0$, on déduit une loi de 'conservation' d'énergie,

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{5}$$

En supposant une équation d'état $P = \omega\rho$, l'intégral de l'équation (5) donne

$$\rho = \rho_i \left(\frac{a_i}{a}\right)^{3(1+\omega)} . \tag{6}$$

Les trois équations (3), (4), (5), dont seulement deux sont indépendantes, sont les équations de base pour un univers Friedmann-Lemaître. Dans l'histoire de l'univers, une phase dominée par radiation, $\omega = 1/3$, a été suivie d'une phase dominée par matière, $\omega = 0$. Dans ces cas le comportement du facteur d'échelle est

$$\begin{aligned} a(\tau) &\propto \tau^{1/2} \quad \text{pour} \quad \omega = 1/3, \\ a(\tau) &\propto \tau^{2/3} \quad \text{pour} \quad \omega = 0, \end{aligned} \tag{7}$$

selon les équations (3) et (6). Les équations (7) décrivent un univers en expansion. L'instant $\tau = 0$ dans le passé, où les sections spatiales étaient d'épaisseur nulle, correspond à une singularité initiale, le big bang. Pour resoudre des problèmes cosmologiques, comme celui de l'horizon et de la platitude, on peut supposer une phase inflationaire après le big bang. Après une durée très courte, celle-ci passe à la phase de radiation. À présent, les observations indiquent, que nous nous trouvons dans une phase d'expansion accélérée, dans laquelle l'énergie est dominée par la constante cosmologique Λ_4,

$$a(\tau) \propto \mathrm{e}^{\sqrt{\Lambda_4/3}\tau}. \tag{8}$$

Dimensions supplémentaires

Du point de vue cosmologique, la singularité initiale est une des raisons les plus importantes de chercher des nouvelles théories fondamentales, qui permettent une description de l'évolution cosmologique sans singularités. À présent le candidat le plus prometteur est la théorie des cordes, qui est née de la physique des particules, lorsqu' on cherchait une théorie pour l'interaction forte. Dans la théorie des cordes, les objets fondamentaux ne sont plus des particules ponctuelles, mais des cordes unidimensionnelles, dont les oscillations font naître un spectre de masse. En particulier, ce spectre contient une particule à masse nulle et spin deux, le graviton. À basses énergies l'action de la théorie des cordes se réduit à l'action de la relativité générale. Ainsi, la théorie des cordes contient la gravitation. Ce fait n'est pas évident du tout et représente un argument fort en faveur de cette théorie. La plupart des conséquences phénoménologiques cependant se manifestent seulement à des très hautes énergies où à des très petites échelles de longueur et ne sont pas (encore?) vérifiées expérimentalement.

Afin d'être consistante, la théorie des cordes doit être supersymmétrique et se dérouler dans un espace-temps à dix dimensions. Elle prédit ainsi l'existence de six dimensions spatiales supplémentaires. D'habitude on suppose, que celles-ci sont compactes et petites, afin d'expliquer, pourquoi elles n'étaient pas observées jusqu'à présent. Remarquons ici que l'idée des dimensions supplémentaires a été introduite déjà à partir de 1914 par Nordström, Kaluza et Klein dans un essai d'unifier la gravitation et l'éléctromagnétisme.

Dans le chapitre sur les dimensions supplémentaires nous expliquons en détail comment on trouve le nombre de dix pour les dimensions de l'espace-temps par un argument d'invariance de Lorentz. Pour passer à un espace-temps à dimension plus basse, on compactifie un nombre de dimensions spatiales. Nous discutons des apects géométriques d'espaces compacts ainsi que la compactification de Hořava et Witten. Celle-ci mène à un espace-temps effectivement cinq-dimensionnel, qui est utilisé fréquemment dans la cosmologie branaire.

Les expériences de la physique des particules testent les interactions de jauge jusqu'à des échelles de $1/200\,\mathrm{GeV}^{-1} \simeq 10^{-15}$ mm. Donc on sait, que des particules comme le photon doivent être liées dans dans notre universe quadridimensionnel. Comme nous allons le voir, cette observation peut être expiquée par la théorie des cordes. Au contraire, la gravitation est sensible au nombre total des dimensions spatiales. Par conséquent on s'attend à ce que la loi de Newton soit changée. En effet, en présence de n dimensions supplémentaires compactes la loi de Newton est

$$F = G_D \frac{\mu M}{r^{2+n}} \tag{9}$$

à des distances r petites devant la largeur L des dimensions supplémentaires. La forme habituelle $F \sim r^{-2}$ est confirmée expérimentalement seulement au-dessus de $20\,\mu$m. Des mesures plus précises de la loi de Newton pourraient ainsi révéler l'existence des dimensions supplémentaires dans une expérience de laboratoire quadridimensionnelle.

Dans l'équation (9) la quantité G_D est la constante de Newton fondamentale, qui donne la 'vraie' grandeur de la gravitation dans un espace-temps à D dimensions. La constante de Newton dans notre univers est une quantité dérivée,

$$G_4 \propto \frac{G_D}{L^n}. \tag{10}$$

Cette relation permet une solution au problème de hierarchie entre les forces de gravitation et de jauge. En supposant que G_D est du même ordre de grandeur que la constante de couplage électro-faible, la hierarchie est enlevée de la théorie fondamentale. Dans la théorie effective quadridimensionnelle la gravitation apparaît beaucoup plus faible que les interactions de jauge, parce que elle est diluée (voir le facteur $1/L^n$) dans les dimensions supplémentaires.

Branes

Dans la cosmologie quadridimensionnelle on connaît des objets étendus comme des cordes cosmiques et des murs de domaine. En généralisant cette observation, un espace-temps D-dimensionnel peut contenir des sous-variéteés Lorentziennes de dimension $p + 1 \leq D$. Ces objets sont nommés p-branes, où p désigne le nombre leurs dimensions spatiales. Dans les théories de supergravité on trouve des p-branes comme solutions solitoniques, et dans la théorie des cordes comme des hypersurfaces contenant les degrés de liberté de jauge.

Dans la cosmologie branaire notre univers est identifié avec une 3-brane, et la matière et les champs de jauge sont restreints sur cette brane. Un observateur sur la brane ne peut percevoir les dimensions supplémentaires que par la gravitation.

Dans le chapitre sur les branes nous introduisons des éléments de la géométrie différentielle, qui seront utiles pour la description géometrique des p-branes. Par exemple, la courbure d'une p-brane par rapport à l'espace-temps D-dimensionnel est décrite par le tenseur de courbure extrinsèque. Dans la cosmologie branaire one considère souvent le cas, où il n'y a effectivement qu'une dimension supplémentaire. Dans ce cas là on peut lier le tenseur de courbure extrinsèque K_{AB} au contenu matériel de la brane S_{AB} par les conditions de raccordement de Israel,

$$K^{>}_{AB} - K^{<}_{AB} = \kappa_5^2 \left(S_{AB} - \frac{1}{3} q_{AB} S \right). \tag{11}$$

Le signes $>$ et $<$ dénotent la valeur de K_{AB} sur les deux côtés de la brane, q_{AB} est la première forme fondamentale et S est la trace de S_{AB}. Cette relation nous permettera de trouver des équations cosmologiques (en analogie avec les équations de Friedmann) sur la brane, qui represente notre univers.

Une brane peut agir comme source d'un champ gravitationnel ainsi que d'un champ de jauge. Nous discutons une géométrie statique crée par une collection de N 3-branes en analogie avec la solution de Reissner-Nordström pour un trou noir d'une masse M est d'une charge $Q \propto N$. Cette géométrie nous sert comme background pour des applications cosmologiques. Dans une certaine limite elle se réduit à un espace-temps anti-de Sitter à cinq dimensions plus une partie sphérique. La métrique de l'AdS_5 s'écrit

$$\mathrm{d}s_5^2 = \frac{r^2}{L^2} \left(-\mathrm{d}t^2 + \delta_{ij}\, \mathrm{d}x^i \mathrm{d}x^j \right) + \frac{L^2}{r^2} \mathrm{d}r^2, \tag{12}$$

où r est la coordonnée de la dimension supplémentaire. La constante L est le rayon de courbure de l'AdS_5 . Nous allons voir une première application de la métrique (12) pour la cosmologie branaire dans le chapitre suivant.

Cosmologie d'une brane test

Dans la suite nous plaçons la 3-brane, qui représente notre univers, dans le sous-espace Minkowski (avec les coordonnées (t, x^1, x^2, x^3)) de la métrique (12). Dans l'espace-temps courbé la brane bouge le long de la direction radiale, ce qui mène à une expansion homogène et isotrope de la brane. Le facteur d'échelle a est proportionnel à la position radiale r de la brane. Cette idée se généralise à d'autres backgrounds provenant de la théorie de supergravité ou des cordes.

Ici l'expansion est due seulement au mouvement de la brane et non pas à son contenu matériel. Ce pour ça, qu'on appelle ce scénario la 'cosmologie mirage' [90]. On travaille dans une approximation, où la back-reaction de la brane sur la géometrie environnante peut être négligée. Donc les équations du mouvement peuvent être trouvées à partir d'une action du type Nambu-Goto. Dans le cas $p = 3$ celle-ci s'écrit[1]

$$S_{D3} = -T_3 \int \mathrm{d}^4\sigma\, \mathrm{e}^{-\Phi} \sqrt{-g} - T_3 \int \mathrm{d}^4\sigma\, \widehat{C}_4. \tag{13}$$

Pour la métrique (12) par exemple, on voit, que l'énergie totale de la brane est conservée selon le théorème de Noether. À l'aide de cette observation, on trouve une équation différentielle, qui décrit le mouvement radial de la brane. En utilisant la relation $a \propto r$, celle-ci se transforme ensuite en une équation de type Friedmann.

Perturbations on a moving D3-brane and mirage cosmology (article)

Ce chapitre correspond à l'article 'Perturbations on a moving D3-brane and mirage cosmology', dans lequel nous étudions l'évolution des perturbations cosmologiques sur une 3-brane en mouvement dans un espace-temps anti-de Sitter-Schwarzschild à cinq dimensions. Tout d'abord on trouve que le mouvement de la brane non-perturbée mène à une expansion homogène et isotrope donné par une équation de type Friedmann,

$$H^2 = \left(\frac{a_\tau}{a}\right)^2 = \frac{1}{L^2}\left[\frac{\tilde{E}^2}{a^8} + \frac{1}{a^4}\left(2q\tilde{E} + \frac{r_H^4}{L^4}\right) + (q^2-1)\right], \tag{14}$$

où $\tilde{E} \equiv E - q\frac{r_H^4}{2L^4}$, et r_H est le rayon de Schwarzschild. Le paramètre q est égal à $+1$ pour une brane dont la masse est égale à sa charge (appelée une brane BPS). Le terme en a^{-8}, dominant à des temps tôts, correspond à un fluide avec l'équation d'état $\omega = 5/3$, tandis que le terme en a^{-4} représente de la radiation dite 'noire', qui ne correspond pas à de la radiation physique. Si la supersymmétrie sur la brane est brisée, $q \neq \pm 1$, l'équation (14) a également une contribution $q^2 - 1$, qui joue le rôle d'une constante cosmologique. Tous ces termes sont dus seulement au mouvement de la brane. La solution de l'équation (14) dans le cas supersymmétrique, $q = \pm 1$, est

$$a(\tau) \sim \tau^{1/4} + \tau^{1/2}. \tag{15}$$

Concernant les perturbations, nous ne voulons étudier également que les effets due au mouvement de la brane. En négligeant la back-reaction, les seules perturbations possibles sont celles par rapport au plongement non-perturbé. Celles-ci peuvent être décrites par un champ scalaire ϕ. Nous dérivons une équation du mouvement pour ϕ, où la géométrie du bulk[2] intervient comme masse effective.

[1] T_3 est la tension de la brane, σ^μ les coordonnées intrinsèques, Φ le dilaton à dix dimensions, g la détérminante de la métrique induite et $\widehat{C}_4$ un champ de jauge Ramond-Ramond.

[2] En général le terme 'bulk' désigne l'ensemble des dimensions de l'espace-temps.

Pour une brane BPS, paramétrisée par une énergie conservée $E = 0$, on trouve pour les modes **superhorizons**

$$\phi_k = A_k a^4 + B_k a^{-3}, \tag{16}$$

où les constantes A_k and B_k sont déterminées par les conditions initiales. Donc les modes superhorizons croissent comme a^4 (a^{-3}), quand la brane est en expansion (contraction). Pour les modes **subhorizons** on trouve

$$\phi_k = A_k \frac{\mathrm{e}^{ik\eta}}{a} + B_k \frac{\mathrm{e}^{-ik\eta}}{a}, \tag{17}$$

où η est le temps conforme sur la brane. Les modes subhorizons sont stables, lorsque la brane est en expansion, mais ils croissent sur une brane en contraction. En particulier, la brane devient instable, quand elle s'approche du trou noir de la géométrie AdS_5-Schwarzschild ($r \searrow 0, a \searrow 0$), parce que tous les modes croissent. Dans l'article nous discutons également les cas $E > 0$ et des branes avec $q \neq 1$.

Si on identifie alors la 3-brane avec notre univers, les perturbations ϕ_k induisent des perturbations cosmologiques scalaires. On peut démonter, que les perturbations du plongement ϕ sont liés directement aux potentiels de Bardeen par

$$\begin{aligned} \Phi &= -\left(\frac{\tilde{E}}{a^4} + q\right)\left(\frac{\phi}{L}\right), \\ \Psi &= 3\Phi + 4q\left(\frac{\phi}{L}\right). \end{aligned} \tag{18}$$

Nous pensons, que ces perturbations sont importantes, également si on inclût de la matière sur la brane. Remarquons aussi, que cette approche peut être généralisée aux cas, que la co-dimension de la brane est plus grande que un.

Cosmologie sur une brane avec back-reaction

Un problème de la cosmologie mirage est certainement, qu'on a négligé la back-reaction de la brane sur la géometrie du bulk. S'il n'y a qu'une dimension supplémentaire (si le nombre des co-dimensions d'une brane est un), on peut tenir compte de la back-reaction grace aux conditions de raccordement (11). Pour cette raison la cosmologie branaire utilise souvent un espace-temps effectivement cinq-dimensionnel, selon une compactification de l'espace-temps dix-dimensionnel proposée par Hořava et Witten. Dans ce scénario la brane est fixée par rapport à la dimension supplémentaire, mais le bulk est dépendent du temps (en contraste avec la cosmologie mirage[3]).

À l'aide des conditions de raccordement, Binétruy *et al.* [18] ont trouvé une équation d'évolution sur la brane,

$$H^2 = \frac{\kappa_5^2}{6}\rho_B + \frac{\kappa_5^4}{36}\rho^2 - \frac{\mathcal{K}}{a^2} + \frac{\mathcal{C}}{a^4}, \tag{19}$$

[3] On peut montrer, que les deux points de vue sont liés par une transformation de coordonnées et donc équivalents.

où κ_5^2 est lié à la constante de Newton cinq-dimensionelle, et ρ_B est la densité d'énergie correspondante à une constante cosmologique dans le bulk. La densité d'énergie sur la brane (d'un fluide parfait quelconque) intervient comme ρ^2, contrairement à ce qu'on trouve dans l'équation de Friedmann standard ($H^2 \propto \rho$). Le terme en a^{-4} correspond à radiation noire et peut être identifié avec le terme a^{-4}, qu'on a trouvé déjà dans l'équation (14). En supposant, que la constante cosmologique dans le bulk est compensée par un terme sur la brane, on peut montrer, que les solutions de l'équation (19) sont de la forme

$$a(\tau) \sim (\tau + \tau^2)^{1/q}, \tag{20}$$

où $q = 3(1+\omega)$. Donc à grands temps le deuxième terme domine et on retrouve le comportement standard pour radiation et matière.

Dans ce chapitre nous discutons également les équations d'Einstein dans un univers branaire [142],

$$\widehat{\mathcal{G}}_{\mu\nu} = -\Lambda_4 g_{\mu\nu} + 8\pi G_4 \tau_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} - E_{\mu\nu}. \tag{21}$$

Ici $\widehat{\mathcal{G}}_{\mu\nu}$ est le tenseur d'Einstein construit à partir de la métrique induite $g_{\mu\nu}$, et $\tau_{\mu\nu}$ est le tenseur d'énergie-impulsion sur la brane. Λ_4 est la constante cosmologique effective. Le terme $\pi_{\mu\nu}$ est quadratique dans $\tau_{\mu\nu}$ (menant au ρ^2 dans l'équation (19)), et $E_{\mu\nu}$ est une projection du tenseur de Weyl dans le bulk. Ce dernier terme represente des ondes gravitationelles à cinq dimensions.

Dynamical instabilities of the Randall-Sundrum model (article)

Randall et Sundrum ont proposé un modèle branaire statique, où la métrique sur la brane est multipliée par un facteur exponentiellement décroissant le long de la cinquième dimension. Ceci permet d'avoir une dimension supplémentaire non-compacte [127] ainsi que de résoudre le problème de la hierarchie [128]. Leur modèle se base sur un fine-tuning entre la constante cosmologique (negative) Λ dans le bulk et la tension V de la brane.

Dans l'article 'Dynamical instabilities of the Randall-Sundrum model' nous étudions une généralisation dynamique de ce modèle. Dans une première partie nous essayons de réaliser ce fine-tuning d'une façon dynamique. Nous montrons, que l'énergie potentielle d'un champ scalaire sur la brane ne peut pas annuler Λ. Ce résultat est généralisé pour toute matière satisfaisant la condition d'énergie faible.

Dans la deuxième partie nous dérivons les équations d'Einstein pour une 3-brane dans un bulk à cinq dimensions. En perturbant ces équations nous pouvons étudier la stabilité du modèle RS. Dans ce but le fine-tuning est perturbé par une constante Ω,

$$V = \sqrt{-12\Lambda}(1 + \Omega). \tag{22}$$

Nous trouvons des solutions analytiques aux équations de perturbation à cinq dimensions dont celle pour le facteur d'échelle sur la brane s'écrit,

$$a^2(\tau) \simeq 1 + 2\mathcal{Q}\tau + 4\alpha^2\Omega\tau^2. \tag{23}$$

Donc une instabilité quadratique dans le temps cosmique τ apparaît. En plus, il y a un mode $\mathcal{Q}$, qui represente une instabilité linéaire en τ, même si la condition de fine-tuning n'est pas perturbée du tout. Comme toutes les équations sont invariantes de jauge, ce mode n'est pas simplement du au choix des coordonnées. On peut montrer, que $\mathcal{Q}$ correspond à la vitesse de la brane, si on relâche la condition, que celle-ci est fixée.

Les anisotropies dans le rayonnement du fond diffus pour des univers branaires

La prochaine démarche est d'étudier les conséquences observables des univers branaires. Le rayonnement du fond diffus (CMB) s'est avéré comme moyen puissant pour tester des modèles cosmologiques, et donc on espère d'obtenir des contraintes sur les modèles branaires à partir de leurs prédictions sur le CMB. Dans ce but la théorie de perturbations cosmologiques à cinq dimensions a été dévéloppée pendant les dernières années [129].

Dans ce chapitre et dans l'article suivant nous nous interessons aux perturbations vectorielles induites dans un univers branaire par des ondes gravitationnelles dans le bulk. Comme nous allons voir, leur comportement est radicalement différent de celui dans la cosmologie standard.

Nous considérons un bulk avec la métrique AdS_5 (voir l'équation (12)), et nous dérivons les équations d'Einstein perturbées. Une 3-brane, qui represente notre univers, est placée ensuite dans la géométrie perturbée. Semblablement à la cosmologie mirage, cette brane est en mouvement, mais cette fois-ci on tient compte de la back-reaction par les conditions de raccordement (11). Celles-ci nous permettent en même temps de trouver les perturbations vectorielles induites sur la brane.

Une fois que les perturbations sur la brane sont connues, on peut procéder en utilisant la théorie standard du CMB. Dans ce cadre nous établissons le lien entre les perturbations vectorielles et les fluctuations de température dans le CMB ainsi que le spectre de puissance associé.

CMB anisotropies from vector perturbations in the bulk (article)

Dans l'article 'CMB anisotropies from vector perturbations in the bulk' nous estimons les anisotopies vectorielles dans un univers branaire. Nous résolvons les équations d'Einstein mentionnées ci-dessus dans le cas le plus général et pour des conditions initiales quelconques. Les perturbations vectorielles sur la brane sont obtenues par les conditions de raccordement. Certaines des solutions montrent une croissance exponentielle dans le temps conforme sur la brane, contrairement aux modes vectoriels dans la cosmologie standard, qui décroissent comme a^{-2} quelques soient les conditions initiales. Les modes croissants sont d'énergie finie et parfaitement normalisables et posent donc un problème sévère pour les univers branaires.

Le fait, que ces modes sont normalisables est du à la structure particulière du bulk. Comme dans la plupart des modèles branaires, on a imposé, que le bulk soit symmétrique sous des reflexions, qui laissent la position radiale de la brane fixe (symmétrie Z_2). Pour voir l'essence de la physique derrière ces modes, considérons l'example suivant: les solutions de l'équation de Klein-Gordon pour une masse négative au carré sont de la forme $\exp[\pm k(r \pm t)]$. En particulier, les solutions $\exp[-k(r-t)]$ avec $r > 0$ sont initialement petites, mais croissent dans le temps, si on ne pose pas l'amplitude initiale à zéro. La situation pour les modes vectoriels est analogue. Dans notre cas la fréquence spatiale Ω des perturbations dans le bulk joue le rôle de la masse au carré. Si elle est négative, il y a des solutions exponentiellement croissantes. En plus de ces modes exponentiels, ils existent des modes, qui croissent seulement comme une loi de puissance du facteur d'échelle, mais qui mènent néanmois à des effects importants dans le CMB.

Nous estimons les anisotropies causées par les modes exponentiellement croissants en calculant analytiquement le spectre de puissance C_ℓ dans certaines approximations. Le fait, que les fluctuation de temperature aujourd'hui sont de l'ordre de 10^{-10} contraint l'amplitude primordiale des modes vectorielles d'être enormément petite:

$$A_0(\Omega) < \mathrm{e}^{-10^3}, \text{ pour } \Omega/a_0 \simeq 10^{-26}\,\mathrm{mm}^{-1}, \tag{24}$$

et

$$A_0(\Omega) < \mathrm{e}^{-10^{29}}, \text{ pour } \Omega/a_0 \simeq 1\,\mathrm{mm}^{-1}, \tag{25}$$

où a_0 est le facteur d'échelle aujoud'hui. Dans l'équations ci-dessus on a demandé un spectre invariant d'échelle (autour de $\ell \simeq 10$), et on a fixé le rayon de courbure de AdS_5 à la valeur $L = 10^{-3}$ mm.

Comme les modes vectoriels dans le bulk sont d'énergie finie, ils peuvent être excités par divers processus, par exemple de l'inflation dans le bulk. S'il n'existe pas de mécanisme, qui interdit leur production, les universe branaires anti-de Sitter ne peuvent pas reproduire une cosmologie homogène et isotrope.

Cosmologie des gas de cordes

Dans les derniers deux chapitres nous présentons une idée, qui utilise une symmétrie intrinsèque de la théorie des cordes pour éviter la singularité initiale. Ce scénario s'appelle la 'cosmologie des gas de cordes' ou simplement la 'cosmologie des cordes'. Il y a un nombre de différences fondamentales par rapport aux univers branaires, qu'on a discuté précédemment.

Supposons que l'espace possède la topologie d'un tore neuf-dimensionnel, sur lequel se propagent des cordes fondamentales. Les états d'une corde fermée sont des états oscillatoires, des états d'impulsion (correspondant au mouvement du centre de masse de la corde) et des états d'enroulement. Ces derniers sont possibles à cause de la topologie toroidale. Chaque état excité contribue à la masse d'une corde selon

$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{\omega R}{\alpha'}\right)^2 + \text{oscillateurs}, \tag{26}$$

où R est le rayon d'une dimension compacte du tore, et n et ω sont des nombres d'excitation. On observe, que la formule (26) est invariante sous la transformation

$$R \to \frac{\alpha'}{R}, \quad n \to \omega, \quad \omega \to n, \tag{27}$$

qui peut être appliquée à chaque direction R du tore. D'après cette symmétrie, appelée la dualité T, les spectres de masse sont les mêmes sur le tore original avec des rayons R et le tore 'dual' avec des rayon α'/R. Dans la théorie des cordes on pense, que tout processus doit satisfaire à cette symmétrie. En particulier, on peut décrire l'évolution de l'univers en termes de R ou en termes de α'/R sans différence pour les résultats physiques. Ainsi on peut éviter de tomber sur un singularité lorque $R \searrow 0$ en considérant la théorie duale où $\alpha'/R \nearrow \infty$. Remarquons que l'équation de Friedmann n'est pas symmétrique sous la transformation $a \to 1/a$ (où $a \propto R$), et la singularité initiale est souvent inévitable.

Pour un gas de cordes on a pu montrer [29] que la température satisfait à

$$T(R) = T\left(\frac{\alpha'}{R}\right). \tag{28}$$

La température reste donc toujours finie, tout en évitant la singularité initiale.

Ce scénario proposé par Brandenberger et Vafa offre également une explication élégante, pourquoi nous vivons dans un espace trois-dimensionnel. Supposons, que initialement toutes les dimensions du tore étaient compactes et petites (de l'ordre de l'échelle des cordes). L'état initial est donc un gas chaud et dense de cordes, et comme condition initiale on demande, que toutes les directions soient en expansion isotrope. Cependant les modes d'enroulement empêchent, que le tore s'aggrandisse. Seulement dans un sous-espace tridimensionnel les modes d'enroulement peuvent s'annihiler, car la probabilité, qu'ils s'y intersectent est non-nulle. Ce sous-espace tridimensionnel peut donc devenir grand et constitue finalement notre univers observable. Les autres six dimensions du tore restent petites, telles qu'elles soient invisibles aujourd'hui.

On T-duality in brane gas cosmology

Cette idée a été généralisée pour le cas où la matière sur le tore contient aussi des p-branes. Dans une géométrie toroidale une p-brane peut avoir des modes d'enroulement en analogie avec une corde fondamentale. Dans l'article 'On T-duality in brane gas cosmology' nous établissons une formule analogue à l'équation (26) pour les différentes p-branes, et nous montrons, que sous la transformation (27) une p-brane devient une (9-p)-brane dont la masse est

$$M^*_{9-p} = M_p\,. \tag{29}$$

Donc de nouveau on trouve les mêmes degrés de liberté dans la théorie originale et duale, condition nécessaire pour que les deux soient équivalentes. Pour prouver l'absence d'une singularité initiale, il faudrait encore montrer, que la relation (28) est valable également pour un gas de p-branes.

Remarquons que dans la cosmologie des gas de branes nous ne vivons pas sur une brane particulière. Le rôle des branes est seulement de régler la dynamique de l'espace-temps. En effet, en généralisant l'argument donné ci-dessus, nous avons montré, que le nombre de dimensions, qui deviennent 'grandes' est également trois.

Dans les modèles des univers branaires, qu'on a étudié dans cette thèse, on a toujours trouvé des instabilités dynamiques. Ceci indique, qu'il est très difficile de construire une cosmologie valable avec ce genre de modèles. D'autre part, le scénario de la cosmologie des gas de branes offre une alternative interessante, même s'il y a encore beaucoup de questions ouvertes. En comparant les deux, il semble finalement, que la cosmologie des gas de branes soit la plus prometteuse pour unifier la théorie des cordes et la cosmologie.

Introduction

Superstring theory is a fundamental theory, which unifies gravity and gauge interactions in a consistent and renormalizable way. The fundamental constituents are no longer point-like particles, but 1-dimensional 'strings', whose oscillations give rise to a spectrum of particles. In particular, this spectrum contains a massless spin-2 state, which is identified with the graviton. It was also shown that the low energy action of string theory reduces to the Einstein-Hilbert action of general relativity. Therefore, string theory includes gravity. At the same time, string theory gauge groups such as E_8 contain $SU(3) \times SU(2) \times U(1)$ as subgroups, and hence the standard model of particle physics can be accommodated.

For consistency requirements, such as Lorentz invariance and anomaly cancellation, superstring theory requires the number of space-time dimensions to be 10. It therefore predicts the existence of six spatial extra-dimensions. Certainly, this is a logical possibility, which however has not been empirically verified until now. In fact, these extra-dimensions could be rolled up to small circles, such that they are visible only at very high energies.

String theory also predicts the existence of (p+1)-dimensional hypersurfaces to which standard model fields are confined. An observer on such a 'p-brane' would be able to notice the presence of extra-dimensions only by gravitational interactions, because gravity is the only fundamental force propagating in the whole 10-dimensional space-time. Aside from the Einstein-Hilbert term, the low energy action of string theory contains also a number of new fields that are not present in the standard model, for instance the dilaton and the axion, which may play an important role in cosmology.

The idea of extra-dimensions has originally been introduced by Nordström, Kaluza, and Klein in order to unify gravity and electromagnetism. Many decades later, the development of supergravity theories led to a revival of this idea. In the framework of supergravity, the presence of seven additional dimensions is required, and p-branes arise naturally as classical solitonic solutions.

The significance of string theory for cosmology is that it can possibly resolve the initial singularity (big bang) problem and, moreover, provide initial conditions. Therefore, it is important to investigate how string theory predictions such as extra-dimensions and branes manifest themselves in a cosmological context. This is the principal aim of this thesis.

String theory is not supposed to modify the cosmological evolution from nucleosynthesis onward, where the physics are quite well understood and agree with observations. But as a theory of quantum gravity, string theory is expected to play an important role near the Planck scale. Stringy physics could have left an imprint in the early universe and is probably necessary to understand it. On the other hand, one hopes to learn more about string theory through cosmological observations.

The main body of this thesis consists of four articles corresponding to chapters 5, 7, 9, and 11. The published versions have been retained unchanged, apart from some adaptations of the notation for overall consistency. The purpose of the remaining chapters is to embed the research carried out in the literature that already exists, as well as to provide a short introduction to each topic. Thesis est omnis divisa in partes tres[4].

Part I: Extra-dimensions and branes

In part I we briefly review the standard cosmology, particularly emphasizing issues which are of direct relevance for the present work. To start with, we point out the reason why string theory and supergravity require $D = 10$ or $D = 11$ space-time dimensions. Then, we work out the compactification of extra-dimensions in general, as well as on a torus and on an orbifold in particular. If extra-dimensions indeed exist, Newton's law would get modified: in the presence of n compact extra-dimensions, it would be a $1/r^{2+n}$ law at scales much smaller that the compactification radius. While the 4-dimensionality of gauge interactions has been tested down to $1/200\,\mathrm{GeV}^{-1} \simeq 10^{-15}\,\mathrm{mm}$, Newton's law is experimentally confirmed only above $L \sim 20\,\mu\mathrm{m}$, thus leaving room for new physics below that scale. This is an example where the scenario of extra-dimensions, and indirectly also sting theory, can be tested even in the laboratory.

Recently, it has been argued that relatively large compact extra-dimensions (*i.e.* with $L \simeq \mu\mathrm{m}$) can solve the hierarchy problem: the effective 4-dimensional Newton constant is given by $G_4 \propto G_D/L^n$, where G_D is the fundamental gravitational constant, which can be of the order of the electroweak scale so that, in the fundamental theory, the gap between the gravitational and the electroweak scale disappears.

Part I is kept rather general and often goes somewhat beyond cosmology in order to put our work into a broader context.

Part II: Brane cosmology

Part II is devoted to brane cosmology. Superstring theory and M theory suggest that our observable universe could be a (3+1)-dimensional hypersurface, a 3-brane, embedded in a 10 or 11-dimensional space-time. This idea has recently received a great deal of interest. Such brane worlds have also been studied earlier in the context of topological defects, before branes were discovered to have a string theory realization.

A natural link between string theory and cosmology can be made within a framework called the mirage cosmology [90]. In this approach, our universe is identified with a probe 3-brane moving in a higher-dimensional space-time, which is given by a supergravity solution. If the bulk metric has certain symmetry properties, the unperturbed brane motion leads to a homogeneous and isotropic expansion or contration with a scale factor $a(\tau)$ on the brane. In the article 'Perturbations on a moving D3-brane and mirage cosmology' in Chap. 5, we study the evolution of perturbations on such a moving brane. Deviations from the unperturbed embedding give rise to perturbations around the Friedmann-Lemaître

[4] C.Iuli Caesaris commentariorum de bello gallico, liber primus.

solution, and those 'wiggles' can be directly related to the gauge invariant Bardeen potentials. We show that on an expanding brane superhorizon modes grow as a^4, while subhorizon modes are stable. On a contracting brane, both super- and subhorizon modes are growing. These perturbations evolve as a consequence of the brane motion only and are not sourced by matter. However, they are expected to be important if matter is also included. Given the probe nature of the brane, our method has many similarities with the study of topological defects. For example, the dynamics and perturbations are derived from the Dirac-Born-Infeld action, which is a generalization of the Nambu-Goto action. Therefore, it is not difficult to apply this method even when the number of co-dimensions is greater than one. On the other hand, we cannot include the back-reaction of the brane onto the bulk geometry, and this is a major shortcoming of the mirage cosmology.

In Chap. 6 and the following, we therefore investigate other brane world models, which accommodate the back-reaction via junction conditions linking the (real) energy content of the brane to the geometry of the bulk. This approach in turn is limited to the case of one co-dimension because, when there are more than one extra dimensions, the junction conditions do not apply anymore. Therefore, much work has focused on the case in which our universe is a 3-brane in a 5-dimensional bulk. This scenario is motivated by Hořava-Witten compactification, where the brane is located at an orbifold fixed point. The bulk is in general time-dependent, and via the junction conditions this leads to a cosmological evolution on the brane. Binetruy *et al.* derived a Friedmann-like equation for the brane world and showed, that the standard evolution can be recovered at late times [18]. A more general approach is to derive equations for the Einstein tensor on a 3-brane, which has been carried out by the authors of Ref. [142].

Randall and Sundrum proposed a bulk geometry in which the metric on the 3-brane is multiplied by an exponentially decreasing 'warp' factor, such that transverse lengths become small at short distances along the fifth dimension [127]. This allows for a non compact extra-dimension without coming into conflict with observational facts. A related model was proposed to solve the hierarchy problem [128]. However, both models rely on a fine-tuning between the brane tension and the cosmological constant in the bulk. In the article 'Dynamical instabilities of the Randall-Sundrum model' in Chap. 7, we construct a dynamical generalization of the RS model, and show that, in a cosmological context, small deviations from fine-tuning lead to runaway solutions. We also formulate a no-go theorem showing that the fine-tuning cannot be obtained by a dynamical mechanism involving a scalar field or a fluid on the brane.

The next step is to derive observational consequences of brane world models. The cosmic microwave background (CMB) anisotropies represent some of the most important cosmological observations. Measurements of the temperature fluctuations in the CMB provide us with a window on the early universe, and are therefore also suited to confirm or rule out various brane world models. To that end, a lot of work has been invested recently to derive gauge invariant perturbation theory in brane worlds with one co-dimension [129].

In the article 'CMB anisotropies from vector perturbations in the bulk' in Chap. 9, we take our universe to be a 3-brane moving in a 5-dimensional anti-de Sitter (AdS_5) bulk. The setup is similar to the one in the mirage cosmology, but this time the back-reaction is taken into account. We consider vector perturbations in the bulk, which are modes of 5-dimensional gravity waves, and we find analytically the most general solution of the perturbed Einstein equations for arbitrary initial conditions. Via the junction conditions, these bulk perturbations induce vector perturbations on the brane, and we find exponentially growing modes, which are nonetheless perfectly normalizable. This differs radically from the usual behavior in the standard cosmology, where vector modes are always decaying. We estimate the effect on the angular power spectrum and discuss new severe constraints for brane worlds.

Part III: Brane gas cosmology

In the last part, we present a scenario called brane gas cosmology. It is also motivated by superstring theory, but differs in many aspects from brane world models. In particular, we are not thought to live on a brane, but rather in the bulk. The topology of the background space-time is that of a nine-torus, and the matter source consists of a gas of strings and p-branes.

Initially, all nine spatial dimensions are small and compact and, by a dynamical decompactification mechanism involving the winding modes of the strings and branes, three dimensions grow large.

The main motivation which led to the development of brane gas cosmology is the initial singularity problem in the standard cosmology. In their original work, Brandenberger and Vafa considered a gas of strings and showed that the T-duality symmetry of string theory can be used to give a singularity-free description of the cosmological evolution [29]. T-duality is a symmetry between large and small scales, and it allows to describe the 'region' near the big bang in terms of low curvature scales. With this symmetry it can be shown that for a string gas the temperature remained always finite in the past.

This idea was later extended to include various p-branes in addition to strings. In the article 'On T-duality in brane gas cosmology' in Chap. 11, we establish the action of T-duality on the states making up the brane gas, and show that the mass spectrum is indeed invariant. Based on this, we claim that the initial singularity can be avoided, also in the case that the matter source consists of a gas of branes.

Notations and conventions

Unless explicitly mentioned, we shall use the following notations and conventions throughout this thesis:

- D denotes the total number of space-time dimensions, while d is the total number of purely spatial dimensions, and n the number of extra-dimensions. Thus $D = 1 + d = 1 + 3 + n$.

- M, N label coordinates of the D-dimensional space-time, μ, ν of a Lorentzian submanifold (*e.g.* a brane), i, j of a Riemannian (mostly Euclidean) submanifold, and I, J of a Riemannian submanifold which represents the space of extra-dimensions.

- It is often convenient to split the coordinates of the D-dimensional space-time as $(x^M) = (t, \vec{x}, x^I) = (t, x^i, x^I)$, where $i = 1, \cdots, p$ and $I = p+1, \cdots, d$. In Chaps. 10 and 11 we write $(x^M) = (t, x^n)$ where $n = 1, \cdots, d$. If the focus is on a single extra-dimension x^d, we shall write $(x^M) = (x^\mu, x^d)$. In the case $D = 5$ we set $x^d = y$ or $x^d = r$.

- $(\sigma^\mu) = (\tau, \sigma^i) = (\tau, \sigma^1, \cdots, \sigma^p)$ denote internal coordinates on a submanifold. On a brane, $\sigma^0 \equiv \tau$ corresponds to cosmic time. In some places, we use conformal time η instead.

- Boldface vectors are always 3-vectors, *e.g.* the direction of observation of CMB photons is indicated by $\mathbf{n}$.

- The metric of the D-dimensional space-time is G_{MN}, and the induced or internal metric on a Lorentzian submanifold is $g_{\mu\nu}$. The two are related by a pull-back or push-forward. For clarity, we sometimes use a hat to stress that a quantity is associated with $g_{\mu\nu}$ rather than with G_{MN}. For example, $\widehat{R}_{\mu\nu\rho\sigma}$ is the induced (or internal) Riemann tensor on a brane.

- The metric signature is $- + \cdots +$. The D-dimensional Minkowski metric is $(\eta_{MN}) = \mathrm{diag}(-1, +1, \cdots, +1)$.

- We use $\equiv$ for definitions, $\simeq$ for approximately equal, $\sim$ for a rough correspondence, and $\propto$ for a proportionality.

- M_D denotes the D-dimensional (fundamental) Planck mass, and M_4 the effective Planck mass in our 4-dimensional universe.

- The D-dimensional cosmological constant is denoted by Λ_D, and the cosmological constant in our 4-dimensional universe by Λ_4. As an exception we use Λ instead of Λ_5 for the 5-dimensional cosmological constant due to its frequent appearance.

We work in units $\hbar = c = k_B = 1$, such that there is only one dimension, energy, which is usually measured in GeV. Then,

$$[\text{energy}] = [\text{mass}] = [\text{temperature}] = [\text{length}]^{-1} = [\text{time}]^{-1}.$$

Part I

EXTRA-DIMENSIONS AND BRANES

Chapter 1

The cosmological standard model

1.1 Isotropy and homogeneity of the observable universe

The cosmological standard model relies on three main observations:

1. **Isotropic expansion of the universe**. In 1929 Hubble discovered that the spectra of most galaxies are redshifted and interpreted this as a Doppler shift[1] resulting from their motion away from us ($z = v/c$). Furthermore, he observed that the escape velocity of galaxies is proportional to their distance, $v = Hd$, where H is Hubble's constant. The value of H today is $70 \, \frac{\mathrm{km}}{\mathrm{s \cdot Mpc}}$.

2. **Isotropy of the cosmic microwave background (CMB) radiation**. In 1965 Penzias and Wilson discovered a uniform background of 'cosmic photons' corresponding to black body radiation of 3 K. In fact, in 1948 Gamov had already predicted the existence of this radiation as left over after the combination of electrons and protons into hydrogen during an earlier hotter phase of the universe. Since that moment the CMB photons have travelled freely through the universe and have cooled down to their present temperature of 2.725 K due to the cosmic expansion.

3. **Abundance of light elements**. The fact that high energies are needed to synthesize elements gives another hint that the early universe must have been hot. (Formation in stars alone would not yield the correct abundances.) Nucleosynthesis calculations in the early universe predict the measured abundances of hydrogen, helium, lithium, and deuterium. Nucleosynthesis also gives indirect evidence that the already the early universe must have been very isotropic.

These three pieces of evidence tell us that our universe has emerged from a very hot and dense state called the big bang. In general relativity, the big bang is an initial singularity. During the subsequent expansion, the universe cooled down such that successively more and more structures (nuclei, atoms, molecules) could form. Small gravitational instabilities gave rise to the large scale structure observed today (solar system, galaxies, galaxy clusters). On scales above roughly 100 Mpc, the distribution of matter becomes isotropic[2]. Assuming that we do not inhabit a preferred position in space, we think of the whole universe as being homogeneous on large scales[3]. The dynamics of the expansion can then be described by highly symmetric solutions of Einstein's equations. They are called Friedmann-Lemaître space-times (sometimes including also Robertson and Walker) after the Russian mathematician Alexander Friedmann, who in 1922 derived his cosmological equations, and the Belgian priest Georges Lemaître, who is regarded as the father of

[1] In the framework of general relativity the red-shift is not interpreted as a Doppler shift, but as the expansion of space-time itself.

[2] This is still small compared to the Hubble radius today, $c/H_0 = 3000 h_0^{-1}$ Mpc.

[3] Nowadays, there are heated debates whether matter has a fractal distribution.

the big bang theory. We introduce this geometry in the next section (following the treatment of Ref. [149]).

1.2 Friedmann-Lemaître space-times

1.2.1 Isotropic manifolds

We start with the definitions of Riemannian and Lorentzian metrics and manifolds.

Definition 1. *A* ***Riemannian metric*** *on a differentiable manifold $\mathcal{M}$ is a covariant tensor field g of order two with the following properties:*
(i) $g(X,Y) = g(Y,X)\ \forall X,Y \in \mathcal{X}(\mathcal{M})$*, where $\mathcal{X}(\mathcal{M})$ is the set of all infinitely many times differentiable vector fields on $\mathcal{M}$.*
(ii) *g is non degenerated at each point of $\mathcal{M}$, i.e. if $g(X,Y) = 0\ \forall Y \implies X = 0$.*
(iii) *The signature of g is $(+,\cdots,+)$.*

Definition 2. *A* ***pseudo-Riemannian metric*** *satisfies* ***(i)*** *and* ***(ii)*** *of the above definition, but the signature of g is $(-,\cdots,-,+,\cdots,+)$.*
The special case $(-,+,\cdots,+)$ is called a Lorentzian metric.

Definition 3. *The pair $(\mathcal{M},g)$ is called a* ***Riemannian, pseudo-Riemannian or Lorentzian manifold*** *according to the type of g defined above.*

Isotropy is defined in the following way:

Definition 4. *A 4-dimensional Lorentzian manifold $(\mathcal{M},g)$ is isotropic with respect to the time-like velocity field v, $g(v,v) = -1$, if at each point q*

$$\{T_q\varphi \,|\, \varphi \in \mathrm{Iso}_q(\mathcal{M}), (T_q\varphi)v = v\} \supseteq SO_3(v). \tag{1.1}$$

Here, $T_q\varphi$ is the map between the two tangent spaces at the points q and $\varphi(q)$ i.e. $T_q\varphi : \mathcal{T}_q\mathcal{M} \to \mathcal{T}_{\varphi(q)}\mathcal{M}$, and $\mathrm{Iso_q}(\mathcal{M})$ is the group of local isometries of $\mathcal{M}$ leaving the point q invariant. $SO_3(v)$ is the group of linear transformations in $\mathcal{T}_q\mathcal{M}$ leaving v invariant and inducing special orthogonal transformations in the space orthogonal to v. Somewhat loosely speaking, definition (1.1) states that a space-time is isotropic if the length and angle preserving maps with v invariant contain the group of rotations. One can then show [148] that locally $\mathcal{M}$ can be foliated into a one-parameter family of spatial hypersurfaces Σ_τ (with parameter τ) which are orthogonal to v. The integral curves of v are geodesics of $(\mathcal{M},g)$, i.e. $\nabla_v v = 0$, and the geodesic distance between two hypersurfaces is constant independent of $q \in \Sigma_\tau$. This allows to identify τ with a 'cosmic time'. Furthermore, one can prove that $(\Sigma_\tau, \gamma_\tau)$ (where γ_τ is the induced metric on Σ_τ) is a space of constant curvature. Then the map $\phi : \Sigma_\tau \to \Sigma_{\tau'}$ induced by the flux ϕ along the integral curves of v satisfies $\phi^*\gamma_{\tau'} = \mathrm{const}\cdot\gamma_\tau$, where ϕ^* is the pull-back associated with ϕ. This means that, in comoving coordinates, the metric tensors on all hypersurfaces Σ_τ are equal up to a 'scale factor'.

One may therefore decompose the metric tensor g on $\mathcal{M}$ as[4]

$$g = -\mathrm{d}\tau^2 + a^2(\tau)\gamma, \tag{1.2}$$

where $\mathrm{d}\tau$ is a 1-form obtained by applying the exterior derivative d on the coordinate function τ, and γ is the metric of a 3-dimensional Riemannian space of constant curvature. The scale factor a depends only on the hypersurface and thus on τ. A space-time $(\mathcal{M}, g)$ with g given by Eq. (1.2) is called a Friedmann-Lemaître space-time. Notice that a manifold which is isotropic at each point q is also homogeneous[5].

We now investigate in more detail the Riemannian manifold $(\Sigma_\tau, \gamma_\tau)$.

1.2.2 Riemannian spaces of constant curvature

In the following, we consider $(\Sigma_\tau, \gamma_\tau)$ for some fixed value of τ and omit the subscript. Let ∇ denote an affine connection on Σ, and X, Y, Z, W vector fields in $\mathcal{X}(\Sigma) \subset \mathcal{T}_q\Sigma$. The curvature is the map

$$\begin{aligned} R : \mathcal{X}(\Sigma) \times \mathcal{X}(\Sigma) \times \mathcal{X}(\Sigma) &\longrightarrow \mathcal{X}(\Sigma) \\ R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) &- \nabla_{[X,Y]}Z. \end{aligned} \tag{1.3}$$

The last term vanishes in a coordinate basis. The Riemann tensor is defined via

$$R(X, Y, Z, W) = g(X, R(Z, W)Y), \tag{1.4}$$

which is the scalar product of the two vectors X and $R(Z, W)Y$. Now let $E \subset \mathcal{T}_q\Sigma$ denote an arbitrary plane in the tangent space $\mathcal{T}_q\Sigma$ and X, Y two orthonormal vectors spanning E. For each plane E, the sectional curvature is defined by

$$\mathcal{K}_q(E) = R(X, Y, X, Y). \tag{1.5}$$

Note, that this expression is independent of the basis of E.

Definition 5. *If, for all points $q \in \Sigma$ and for all planes $E \subset \mathcal{T}_q\Sigma$, the sectional curvature $\mathcal{K}_q(E)$ is equal to a constant $\mathcal{K}$, then Σ is called a* ***space of constant curvature****. In the following, we consider $\mathcal{K}$ to be normalized, such that $\mathcal{K} = +1, 0, -1$ for positive, negative and zero curvature.*

Spaces of constant curvature are frequently used also in brane cosmology, and we shall discuss them in more detail in Sec. 2.4. In the next paragraph, we give several coordinate expressions for Σ.

[4]Often g is also denoted by $\mathrm{d}s^2$.

[5]A manifold $(\mathcal{M}, g)$ is called homogenous if locally $g = -\mathrm{d}\tau^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$, in a coordinate basis (τ, x^i). The requirement of homogeneity is weaker than that of isotropy around each point.

1.2.3 The metric of Friedmann-Lemaître space-times

Let us introduce coordinates x^1, x^2, x^2 on Σ. A basis of the co-tangent space $\mathcal{T}_q^*\mathcal{M}$ is then given by the 1-forms $(\mathrm{d}x^0 \equiv \mathrm{d}\tau, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3)$, and the metric tensor on $\mathcal{M}$ can be developed as

$$g = \frac{1}{2}\, g_{\mu\nu}\,(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu) \equiv g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu, \quad g_{\mu\nu} = g_{\nu\mu}. \tag{1.6}$$

In a space of constant curvature one can always choose x^1, x^2, x^3, such that the metric (1.2) takes the form

$$g = -\mathrm{d}\tau^2 + a^2(\tau)\frac{1}{(1+\mathcal{K}\varrho^2/4)^2}\left[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\right], \tag{1.7}$$

where $\varrho^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. Note that, unlike Σ, the manifold $\mathcal{M}$ is in general not a space of constant curvature. On going to polar coordinates one has

$$(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 = \mathrm{d}\varrho^2 + \varrho^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{1.8}$$

and by defining a new radial coordinate

$$r = \frac{\varrho}{1+\mathcal{K}\varrho^2/4}, \tag{1.9}$$

one can rewrite Eq. (1.7) in the form

$$g = -\mathrm{d}\tau^2 + a^2(\tau)\left[\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]. \tag{1.10}$$

Finally, we can substitute

$$r = \begin{cases} \sin\chi & \mathcal{K} = +1, \\ \chi & \mathcal{K} = 0, \\ \sinh\chi & \mathcal{K} = -1, \end{cases} \tag{1.11}$$

to get the form

$$g = -\mathrm{d}\tau^2 + a^2(\tau)\left[\mathrm{d}\chi^2 + \Sigma^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{1.12}$$

where

$$\Sigma^2(\chi) = \begin{cases} \sin^2\chi & \mathcal{K} = +1, \\ \chi^2 & \mathcal{K} = 0, \\ \sinh^2\chi & \mathcal{K} = -1. \end{cases} \tag{1.13}$$

1.3 The gravitational field equations

In general relativity the metric $g_{\mu\nu}$ is a dynamical variable describing the gravitational field. Its dynamics are governed by Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R + \Lambda_4 g_{\mu\nu} \equiv \mathcal{G}_{\mu\nu} + \Lambda_4 g_{\mu\nu} = \kappa_4^2 T_{\mu\nu}, \tag{1.14}$$

which relate the energy-momentum tensor $T_{\mu\nu}$ to the curvature of space-time encoded in the Einstein tensor $\mathcal{G}_{\mu\nu}$. The latter is a combination of the Ricci tensor $R_{\mu\nu}$ and the Riemann scalar R. The strength of the coupling is given by the constant κ_4^2, which is related to the Planck mass M_4 and the Newton constant G_4 by

$$\kappa_4^2 = \frac{1}{M_4^2} = 8\pi G_4. \tag{1.15}$$

This is the only free parameter in general relativity. Finally, the quantity Λ_4 is the cosmological constant.

1.3.1 Cosmological equations

From the gravitational field equations (1.14) one can derive cosmological equations. This is most readily done in an orthonormal basis

$$\omega^0 = \mathrm{d}\tau, \quad \omega^i = \frac{a(\tau)}{1 + \mathcal{K}\varrho^2/4}\,\mathrm{d}x^i. \tag{1.16}$$

Then the components of the Einstein tensor, constructed from the Friedmann metric (1.7), read

$$\begin{aligned} \mathcal{G}_{00} &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{\mathcal{K}}{a^2}\right), \\ \mathcal{G}_{11} = \mathcal{G}_{22} = \mathcal{G}_{33} &= -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\mathcal{K}}{a^2}, \end{aligned} \tag{1.17}$$

where the dot denotes a derivative with respect to cosmic time τ.

On scales where the universe is isotropic and homogeneous ($\gtrsim 100\,\mathrm{Mpc}$), matter can be regarded as a continuous medium, and we assume this to be a perfect fluid. In the basis (1.16) the energy-momentum tensor then takes the form

$$T_{00} = \rho, \quad T_{ij} = P\delta_{ij}. \tag{1.18}$$

With these expressions, the 00 component of (1.14) leads to the first Friedmann equation

$$\frac{\dot{a}^2}{a^2} + \frac{\mathcal{K}}{a^2} = \frac{8\pi G_4}{3}\rho + \frac{\Lambda_4}{3}, \tag{1.19}$$

giving the expansion rate $\dot{a}$ of the universe as a function of its energy content, the spatial curvature and the cosmological constant. It is a first order differential equation, because the 00 Einstein equation is a constraint.

One defines the Hubble parameter by

$$H = \frac{\dot{a}}{a}, \tag{1.20}$$

in analogy to the Hubble constant, which is the ratio v/d. In general, H depends on time.

From the 11 component of Einstein's equations, and by using Eq. (1.19), one obtains the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_4}{3}(\rho + 3P) + \frac{\Lambda_4}{3}. \tag{1.21}$$

Notice that Eqs. (1.19) and (1.21) are relations between measurable quantities and must therefore be independent of the basis or of the metric signature.

1.3.2 Energy 'conservation'

The Einstein tensor satisfies geometrical identities

$$\nabla_\nu \mathcal{G}^{\mu\nu} = 0, \tag{1.22}$$

which are called contracted Bianchi identities. Via Einstein's equations and for $\Lambda_4 = 0$, one has

$$\nabla_\nu T^{\mu\nu} = 0. \tag{1.23}$$

This is, however, not an energy conservation law, as it is not of the form $\partial_\nu T^{\mu\nu} = 0$. Indeed, this non conservation is due to the fact that matter exchanges energy with the gravitational field. But for the gravitational field there is no energy-momentum tensor: at any point $q \in \mathcal{M}$, it can be transformed away, as locally $g_{\mu\nu}(q) = \eta_{\mu\nu}$ and $\Gamma^\mu{}_{\nu\lambda}(q) = 0$, and without field there is no energy nor momentum. Notice also that in special relativity the conservation laws for energy and momentum are based on the invariance of a closed system under translations in space and in time. On a curved manifold, however, such translations are in general not symmetries anymore, and therefore there exists no general energy-conservation law in general relativity[6]

For the Friedmann-Lemaître metric (1.7), the $\nu = 0$ component of Eq. (1.23) yields

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{1.24}$$

giving the local rate of change of the energy density in a Friedmann-Lemaître space-time. This equation is nevertheless called 'the energy conservation law'.

The three equations (1.19), (1.21), and (1.24) are the basic equations governing the dynamics of a Friedmann-Lemaître universe. Only two of them are independent. For example, by using the 'conservation' law (1.24), the second order equation (1.21) can be integrated to obtain the first order equation (1.19).

1.3.3 Past and future of a Friedmann-Lemaître universe

Without solving explicitly the Friedmann equations, we can already make some qualitative statements about the dynamics of a Friedmann-Lemaître universe. Consider Eq. (1.21) with $\Lambda_4 = 0$. As long as $\rho + 3P$ is positive, $\ddot{a}$ must be negative.

[6]If however a Killing field K exists, *i.e.* a field satisfying $\mathcal{L}_K g = 0$, one can construct conserved quantities $T^{\mu\nu}K_\nu$, provided that Eq. (1.23) holds.

Since the scale factor a is positive, and $\dot{a}$ is positive (due to the observed Hubble expansion), $a(\tau)$ is a concave function of τ. Therefore, a must have been zero at some finite time in the past. This singularity of the Friedmann-Lemaître universe corresponds to the big bang. From equation (1.19) one can show, using similar arguments, that when $\Lambda_4 = 0$ the future behavior is entirely determined by the curvature of the spatial sections. For $\mathcal{K} = -1$ and $\mathcal{K} = 0$ the universe expands eternally, while for $\mathcal{K} = +1$ the expansion stops and turns into a contraction. The final fate of the universe is then a 'big crunch' singularity where again $a = 0$.

Alternatively, the qualitative behavior can be obtained by specifying the energy density ρ with respect to a critical energy density ρ_c. For $\Lambda_4 = 0$ the Friedmann equation (1.19) yields

$$\rho = \frac{3}{8\pi G_4}\left(H^2 + \frac{\mathcal{K}}{a^2}\right). \tag{1.25}$$

The spatial curvature $\mathcal{K}$ is positive or negative, according to whether ρ is greater or less than the critical density

$$\rho_c \equiv \frac{3H^2}{8\pi G_4}. \tag{1.26}$$

Therefore, the universe eternally expands for $\rho \leq \rho_c$ and collapses for $\rho > \rho_c$. The value of ρ_c today is

$$\rho_c = 1.88 \cdot 10^{-29}\, h_0^2\, \frac{\mathrm{g}}{\mathrm{cm}^3}, \quad H_0 = 100\, h_0\, \frac{\mathrm{km}}{\mathrm{s}\cdot\mathrm{Mpc}}, \quad h_0 = 0.70, \tag{1.27}$$

where H_0 denotes the Hubble parameter today.

It is useful to measure energy densities in terms of the critical energy density by introducing the dimensionless parameter

$$\Omega_X = \frac{\rho_X}{\rho_c}, \tag{1.28}$$

where X labels a particular contribution or particle species. Here, X will refer to the curvature $\mathcal{K}$ or to the cosmological constant Λ_4, and it will be omitted for the usual matter density ρ. To write to the Friedmann equation in terms of Ω_X, we divide equation (1.19) by H^2,

$$1 = -\frac{\mathcal{K}}{a^2H^2} + \underbrace{\frac{8\pi G_4}{3H^2}}_{=\frac{1}{\rho_c}}\rho + \underbrace{\frac{8\pi G_4}{3H^2}}_{=\frac{1}{\rho_c}}\underbrace{\frac{\Lambda_4}{8\pi G_4}}_{=\rho_{\Lambda_4}}, \tag{1.29}$$

where we now have included Λ_4 and attributed an energy density ρ_{Λ_4} to it, such that finally

$$1 = \Omega_{\mathcal{K}} + \Omega + \Omega_{\Lambda_4}. \tag{1.30}$$

This form of the Friedmann equation is particularly useful for cosmological parameter estimation.

Let us remark here, that the qualitative behavior of cosmological models is often determined by rather general requirements on the energy-momentum tensor $T_{\mu\nu}$, without need to specify a particular matter model. Consider, for example, matter satisfying

$$\left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} T\right) v^\mu v^\nu \geq 0, \tag{1.31}$$

where v is an arbitrary unit time-like 4-vector and $T = T_{\mu\nu} g^{\mu\nu}$ is the trace of the energy-momentum tensor. For a perfect fluid this corresponds to $\rho + 3P \geq 0$. The condition (1.31) is called strong energy condition. Via Einstein's equations (for $\Lambda_4 = 0$),

$$R_{\mu\nu} = \kappa_4^2 \left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} T\right), \tag{1.32}$$

the strong energy condition is equivalent to

$$R_{\mu\nu} v^\mu v^\nu \geq 0 \Longleftrightarrow Ric(v, v) \geq 0. \tag{1.33}$$

Now, if the Ricci tensor of a manifold $(\mathcal{M}, g)$ satisfies this condition, one can prove that, a finite time back in the past, $\mathcal{M}$ is geodesically incomplete, and hence there must have been an initial singularity (for a textbook treatment of these so-called singularity theorems see *e.g.* [160]).

It is however likely that Einstein's equations do not hold near the singularity, and then the strong energy condition (1.31) does not translate into the condition (1.33) for the singularity theorems[7]

Let us mention that there exists also a 'weak energy condition', $T_{\mu\nu} v^\mu v^\nu \geq 0$, which means that the energy density of matter as measured by an observer with 4-velocity v^μ has to be greater or equal than zero. For a perfect fluid, this amounts to $\rho \geq 0$. Finally, the 'dominant energy condition' requires that $-T^\mu{}_\nu v^\nu$ be a future directed time-like or null vector. Physically, this quantity corresponds to the energy-momentum 4-current density of matter as seen by an observer with velocity v. For a perfect fluid, this reduces to $\rho \geq |P|$.

1.3.4 Cosmological solutions

The cosmological equations (1.19), (1.21), and (1.24) can be easily integrated, when the pressure is related to the energy density by an equation of state $P = \omega\rho$. First, the energy 'conservation' law (1.24) gives

$$\rho = \rho_i \left(\frac{a_i}{a}\right)^{3(1+\omega)}, \tag{1.34}$$

[7]In fact, one of the main motivations to introduce stringy physics in cosmology is to avoid the initial singularity. We are presenting a possibility to resolve the singularity problem in the article 'On T-duality in brane gas cosmology' in Chap. 11.

where a_i and ρ_i are determined by the initial conditions. In the following we take them to be the values today, $a_i = a_0$. In the matter dominated era one has $\omega = 0$ and $\rho \sim a^{-3}$, whereas in the radiation dominated era $\omega = 1/3$ and $\rho \sim a^{-4}$. The extra factor a^{-1} is due to the stretching of wavelengths during the cosmic expansion.

Let us assume that $\mathcal{K} = 0$ and $\Lambda_4 = 0$, and integrate the Friedmann equation (1.19) by inserting the solution (1.34) for ρ. This leads to the relation

$$H^2 = H_0^2 \left(\frac{a_0}{a}\right)^{3(1+\omega)} \tag{1.35}$$

with $H_0^2 = \frac{8\pi G_4}{3}\rho_0$. Eq. (1.35) is solved by

$$\frac{a}{a_0} = \left[\left(\frac{a_i}{a_0}\right)^{\frac{3}{2}(1+\omega)} + \frac{3}{2}(1+\omega)H_0(\tau - \tau_i)\right]^{\frac{2}{3(1+\omega)}}, \tag{1.36}$$

where τ_i is some initial time, and $a_i = a(\tau_i)$. For $\omega = 0$, $a \sim \tau^{2/3}$, whereas for $\omega = 1/3$, $a \sim \tau^{1/2}$. It will be interesting to compare these solutions with those obtained in brane cosmology in paragraph 6.3.3.

1.3.5 A remark on conformal time

Sometimes it is convenient to work with a different time coordinate, namely conformal time η, defined via

$$\begin{aligned} g &= -\mathrm{d}\tau^2 + a^2(\tau)\gamma \\ &\equiv a^2(\eta)(-\mathrm{d}\eta^2 + \gamma). \end{aligned} \tag{1.37}$$

The Hubble parameter in conformal time is

$$\mathcal{H} = \frac{a'}{a} = Ha, \tag{1.38}$$

where the prime denotes a derivative with respect to η. The Friedmann equations (1.19) and (1.21) take the form

$$\mathcal{H}^2 + \mathcal{K} = \frac{8\pi G_4}{3}a^2\rho + \frac{a^2\Lambda_4}{3}, \tag{1.39}$$

$$\mathcal{H}' = -\frac{4\pi G_4}{3}a^2(\rho + 3P) + \frac{a^2\Lambda_4}{3}, \tag{1.40}$$

and the energy ‘conservation’ law is

$$\rho' + 3\mathcal{H}(\rho + P) = 0. \tag{1.41}$$

The relation corresponding to Eq. (1.35) is

$$\mathcal{H}^2 = \mathcal{H}_0^2 \left(\frac{a_0}{a}\right)^{1+3\omega}, \tag{1.42}$$

and the solution to that equation is

$$\frac{a}{a_0} = \left[\left(\frac{a_i}{a_0}\right)^{\frac{1}{2}(1+3\omega)} + \frac{1}{2}(1+3\omega)\mathcal{H}_0(\eta - \eta_i)\right]^{\frac{2}{1+3\omega}} . \tag{1.43}$$

Here, η_i is some initial time and $a_i = a(\eta_i)$. For $\omega = 0$, one has $a \sim \eta^2$, and for $\omega = 1/3$, one has $a \sim \eta$.

1.4 The cosmic microwave background

At the beginning of this chapter we mentioned that small gravitational fluctuations in the early universe gave rise to the formation of large scale structures. Similarly, small perturbations in the matter density at the time when the universe was 300000 years old, result in temperature fluctuations in the cosmic microwave background (CMB) radiation today. Since those fluctuations are of the order of 10^{-5}, the universe must have been very isotropic at the age of 300000 years.

In this section, we give a brief description of the origin of the CMB. Historically, its existence was predicted by Gamov in 1946 and was confirmed by Penzias and Wilson in 1965, providing strong support for the idea of a hot and dense beginning of the universe.

In a hot and dense initial phase of the universe, the photons were tightly coupled to baryons and leptons, for example by Thompson scattering[8] and other collision processes. Thereby, the baryons play the role of the 'walls of a cavity' with temperature T. Consequently, the spectral distribution of the photons is that of black body radiation with temperature T.

When T has dropped to about 3000 K (corresponding to an age of 300000 years) the electrons can combine[9] with protons to form hydrogen atoms, and the free electron fraction suddenly drops below 10^{-4}. Consequently, the photon-electron interaction rate, $\Gamma \sim n_e \sigma_T$, (n_e is the free electron density, and σ_T is the Thompson cross section) becomes small compared to the expansion scale, $\Gamma \ll H$, and the photons propagate freely through the universe. Conventionally, one defines a 'last scattering surface' by the condition $n_e = 1/2$ (rather than some constant time parameter).

During the subsequent cosmic expansion, the photons maintain their black body spectrum, but their temperature decreases according to $T \sim a^{-1}$ from 3000 K down to 2.73 K. The spectrum observed today is the most perfect black body spectrum ever measured. Since its maximum is at micrometer wavelengths, the ensemble of photons is called cosmic microwave background. Because those photons have hardly interacted since their moment of emission, the CMB provides us with an image of the universe when it was 300000 years old,

[8] the scattering between photons and non relativistic electrons

[9] Almost universally, the term 'recombine' is used, which is however misleading: the combination happens just once.

If the universe was completely isotropic at last scattering, photons incident from different direction in the sky would have exactly the same temperature. However, this is not quite the case: various experiments have measured small temperature fluctuations in the CMB, called CMB anisotropies. The provide us with rich information on cosmological parameters. We establish the link between matter perturbations and CMB anisotropies in Chap. 8, and apply this formalism to anti-de Sitter brane worlds in Chap. 9.

1.5 Appendix

In this appendix, we fix our conventions (following [150] and [111]), in particular those for the Riemann and Ricci tensors, as they are often a source of sign confusion. Let $(\mathcal{M}, g)$ be a 4-dimensional Lorentzian manifold and let

$$\begin{aligned} x : U \subset \mathcal{M} &\longrightarrow V \subset \mathbb{R}^4 \\ p \in U &\longmapsto x^\mu(p) \in V, \quad \text{with} \quad q \longmapsto 0 \end{aligned} \tag{1.44}$$

be a map from an open neighborhood U of $q \in \mathcal{M}$ to an open neighborhood V of zero in $\mathbb{R}^4$. The set $(x^\mu) = (x^0, x^1, x^2, x^3)$ are the coordinates of the point q. A basis of the tangent space $\mathcal{T}_0\mathbb{R}^4$ is given by the derivations

$$(\partial_\mu) = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right), \tag{1.45}$$

and a basis of the co-tangent space $\mathcal{T}_q^*\mathcal{M}$ is given by the 1-forms

$$(\mathrm{d}x^\mu) = (\mathrm{d}x^0, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3). \tag{1.46}$$

Since $\mathcal{T}_q\mathcal{M}$ is isomorphic to $\mathcal{T}_0\mathbb{R}^4$, the vectors (1.45) are also a basis of $\mathcal{T}_q\mathcal{M}$, and analogously for the co-tangent space.

Let ∇ denote an affine connection on $\mathcal{M}$. The link with the covariant derivative ∇_δ in the direction ∂_δ is

$$(\nabla X)(\mathrm{d}x^\gamma, \partial_\delta) = \langle \mathrm{d}x^\gamma, \nabla_\delta X \rangle, \tag{1.47}$$

where $X \in \mathcal{T}_q\mathcal{M}$ and $\langle\ \rangle$ denotes the contraction. Notice that ∇X is a 1-times covariant and 1-time contravariant tensor, whereas $\nabla_\gamma X$ is a vector. The connection 1-forms $\omega^\alpha{}_\beta$ and the Christoffel symbols $\Gamma^\alpha{}_{\beta\delta}$ are defined by

$$\nabla_\delta \partial_\beta = \omega^\alpha{}_\beta(\partial_\delta)\partial_\alpha = \Gamma^\alpha{}_{\beta\delta}\partial_\alpha, \tag{1.48}$$

thus measuring the 'twist' of the basis vectors as they are transported on the curved manifold $\mathcal{M}$. The Christoffel[10] symbols are the coefficients of the connection 1-forms

$$\omega^\alpha{}_\beta = \Gamma^\alpha{}_{\beta\gamma}\mathrm{d}x^\gamma, \tag{1.49}$$

[10] If the first index is downstairs (upstairs), they are called Christoffel symbols of the first (second) kind.

because the basis (1.45) and (1.46) are dual to each other: $\langle \mathrm{d}x^\gamma, \partial_\delta \rangle = \mathrm{d}x^\gamma(\partial_\delta) = \partial_\delta x^\gamma = \delta^\gamma_\delta$. Furthermore, in the coordinate basis (1.45), the Christoffel symbols can be expressed in terms of metric derivatives as

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda}(g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda}), \tag{1.50}$$

where the subscript $,\gamma$ is an abbreviation for the partial derivative ∂_γ. Notice, that the Christoffel symbols are not components of some tensor.

We have already defined the curvature in Eq. (1.3) as well as the Riemann tensor in Eq. (1.4). The components of the Riemann tensor are

$$R_{\alpha\beta\gamma\delta} = g(\partial_\alpha, R(\partial_\gamma, \partial_\delta)\partial_\beta). \tag{1.51}$$

Alternatively, this can be written as a contraction

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} &= \langle \mathrm{d}x^\alpha, R(\partial_\gamma, \partial_\delta)\partial_\beta \rangle = \langle \mathrm{d}x^\alpha, (\nabla_\gamma \underbrace{\nabla_\delta \partial_\beta}_{=\Gamma^\lambda{}_{\beta\delta}\partial_\lambda} - \nabla_\delta \underbrace{\nabla_\gamma \partial_\beta}_{=\Gamma^\lambda{}_{\beta\gamma}\partial_\lambda})\rangle \\ &= \Gamma^\alpha{}_{\beta\delta,\gamma} + \Gamma^\lambda{}_{\beta\delta}\Gamma^\alpha{}_{\lambda\gamma} - (\gamma \leftrightarrow \delta), \end{aligned} \tag{1.52}$$

where in the first line, we have used the fact that in a coordinate basis the commutator $[\partial_\gamma, \partial_\delta]$ vanishes. The Riemann tensor has the following symmetries (the square brackets denote antisymmetrization):

$$\begin{array}{ll} R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} & \text{Antisymmetry in the first two indices,} \\ R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} & \text{Antisymmetry in the last two indices,} \\ R_{\alpha[\beta\gamma\delta]} = 0 & \text{Vanishing of antisymmetric parts,} \\ R_{\alpha\beta[\gamma\delta;\varepsilon]} = 0 & \text{Bianchi identity.} \end{array} \tag{1.53}$$

In the last equation, the semicolon denotes ∇_ε. Eqs. (1.53) are the complete set of symmetries of the Riemann tensor. From these, one can deduce the additional symmetries

$$\begin{array}{ll} R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} & \text{Symmetry under pair exchange,} \\ R_{[\alpha\beta\gamma\delta]} = 0. & \end{array} \tag{1.54}$$

The Ricci tensor is defined as the contraction of the Riemann tensor on the first and third (or second and fourth) indices,

$$R_{\beta\delta} = g^{\alpha\gamma} R_{\alpha\beta\gamma\delta} = R^\alpha{}_{\beta\alpha\delta}, \tag{1.55}$$

and is symmetric in its indices. From Eq. (1.52), the Ricci tensor in terms of the Christoffel symbols reads

$$R_{\beta\delta} = \Gamma^\alpha{}_{\beta\delta,\alpha} + \Gamma^\lambda{}_{\beta\delta}\Gamma^\alpha{}_{\lambda\alpha} - \Gamma^\alpha{}_{\beta\alpha,\delta} - \Gamma^\lambda{}_{\beta\alpha}\Gamma^\alpha{}_{\lambda\delta}. \tag{1.56}$$

The extension of these definitions to a D-dimensional space-time is straightforward. We shall need some of these expressions in Chap. 8, when we calculate the perturbed Einstein equations in an AdS_5 background.

Chapter 2

Extra-dimensions

2.1 Motivation

The idea that a higher-dimensional approach may help to understand (3+1)-dimensional phenomena appeared several times in the history of physics. In the twenties Theodor Kaluza [87] and Oskar Klein [94] proposed a 5-dimensional theory to unify gravity and electromagnetism. They observed that the 4-dimensional gravitational and electromagnetic fields can be understood as components of the five-dimensional metric tensor. Unfortunately, it turned out that the **Kaluza-Klein** mechanism is not a phenomenologically working theory, but it still serves as a prototype for many higher-dimensional models (see Sec. 2.5).

Until today, there is no experimental evidence for the existence of extra-dimensions. Nevertheless, theories with extra-dimensions were and are developed, because a higher-dimensional approach is often useful to understand 'unconnected' phenomena from a more fundamental and unified perspective. For instance, the existence of extra-dimensions would opened up new possibilities to connect particle physics and cosmology, as well as to answer questions on renormalizability, the unification of forces, and the big bang singularity. Let us take a closer look at this, proceeding cronologically.

In the seventies an 11-dimensional theory called **supergravity** was constructed. Until then, all attempts to unify general relativity and quantum field theory turned out to be non renormalizable. The hope in supergravity was that the divergence problems of quantum gravity theories could be solved if supersymmetry is added. The peculiar dimensionality 'eleven' arises, as it is the largest possible number of space-time dimension (and thus incorporates the largest possible symmetry group) in which a consistent field theory can be constructed. However, these hopes were dashed when it became clear that local supersymmetry is not enough to overcome the divergence problems. Nevertheless, supergravity remains important since it was discovered, later on, to be the low energy limit of string theory. As the supergravity equations of motion are needed in the article 'Perturbations on a moving D3-brane and mirage cosmology' in Chap. 5, we introduce them in Sec. 2.2.

Nowadays the most promising theory of quantum gravity is **string theory**. Its fundamental constituents are no longer point particles, but 1-dimensional objects called strings. They are characterized by a tension, and their excitations give rise to states representing various massless and massive particles, notably the graviton. Supersymmetric string theories require the number of space-time dimensions to be ten. The success of string theory in unifying gravity and gauge interactions without divergences is the main reason today to take extra-dimensions seriously. Some basic ideas of string theory are presented in Sec. 2.3. In particular, it is demonstrated how string theory predicts the number of space-time dimensions, and what the spectrum of a closed string looks like. The latter is relevant for the article 'On T-duality in brane gas cosmology' in Sec. 11.

In all these theories there is a number n of spatial extra-dimensions in addition to the observed three. One can think of this as attaching an n-dimensional space at each point of our 4-dimensional space-time. The total number of space-time dimensions is then $D = 1 + d = 1 + 3 + n$. In the previous examples: $D = 5$ (Kaluza-Klein), $D = 11$ (supergravity), and $D = 10$ (superstring theory).

Why have these extra-dimensions not been observed (yet)? A simple explanation is that they could be curled up to small circles of the order of the Planck length, and that therefore they are 'visible' only at energies near the Planck mass $\sim 10^{19}$ GeV. There exist also scenarios in which this 'compactifiaction scale' is around 1 TeV, such that it could be accessible in future collider experiments.

A rather intuitive explanation of this is the following: think of a thin and lengthy object, for example a pencil. If it is looked at from very far, the pencil appears as a 1-dimensional object, because the eye cannot discern the thickness. But looking from a small distance, the second rolled up dimension of the pencil becomes visible. In physics, 'looking close' means 'looking at high energies', and therefore compact extra-dimensions can only be seen at high-energies.

We make some general remarks about compact spaces in Sec. 2.4, and present two specific types of compactifications, namely toroidal compactification and Hořava-Witten compactification, in section 2.5 and in paragraph 2.5.3.

The presence of extra-dimensions would modify Newton's law of gravitational attraction. Instead of the $F \sim 1/r^2$ form, one finds that the force depends on the number n of extra-dimensions according to $F \sim 1/r^{2+n}$. This is encouraging, for it allows to test the scenario of extra-dimensions in the laboratory without resorting to high energy experiments. A derivation of the modified Newton's law is given in Sec. 2.6.

Finally, in Sec. 2.7 we discuss an idea to solve the hierarchy problem with 'large' extra-dimensions.

2.2 Supergravity

In general relativity the (local) symmetry group is the Poincaré group. The corresponding set of generators can be extended to include supersymmetry transformations. One then obtains a classical field theory called supergravity. In order to make the symmetry group as large as possible one tries to construct a theory with the highest possible number of space-time dimensions. This turns out to be eleven, since for $D > 11$ the spectrum contains massless particles of spin greater than two, and hence is inconsistent already on the classical level. Therefore, from a theoretical point of view, 11-dimensional supergravity is the 'most general' starting point for a unifying theory of gravity and gauge interactions.

Instead of the original 11-dimensional action of supergravity we shall write down its form after a dimensional reduction to $D = 10$. We postpone explaining how to do this until Sec. 2.5, and simply give the result here. The 10-dimensional reduced supergravity action is particularly useful because supergravity is the low energy limit of type IIB super string theory. Obtaining one action from the other by a simple dimensional reduction is possible as the supersymmetry algebras of the two theories are the same.

The low energy action of type IIB superstring theory is[1] [125]

$$
\begin{aligned}
S_{10} = &\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G}\mathrm{e}^{-2\Phi}\left(R + 4(\nabla_M\Phi)(\nabla^M\Phi) - \frac{1}{2}\frac{1}{3!}H_{ABC}H^{ABC}\right)\\
&-\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G}\frac{1}{2}\frac{1}{5!}F_{ABCDE}F^{ABCDE},
\end{aligned}
\tag{2.1}
$$

where κ_{10}^2 is related to the 10-dimensional gravitational constant. The indices are pulled up and down by the 10-dimensional metric G_{MN}. The field Φ is the dilaton[2], and the 3-form field strength H_{ABC} derives from an antisymmetric tensor field B_{MN} via the exterior derivative $H_3 = \mathrm{d}B_2$. The fields in the first line are multiplied by a factor $\mathrm{e}^{-2\Phi}$ which is the inverse squared of the string coupling constant g_s. The power -2 corresponds to the coupling in a tree-level diagram in string theory. Furthermore, there is a 5-form field strength F_{ABCDE} associated with a 4-form potential C_{MNRS} via $F_5 = \mathrm{d}C_4$. The reason why this term is interesting is that the 4-form naturally couples to a 3-brane[3].

We are interested only in certain features of the low energy action that we will use for cosmology, and for simplicity we have omitted a 1-form F_1, a 3-form F_3 as well as fermionic and Chern-Simons terms (gauge invariant terms where wedge products of the p-form potentials appear rather that their derivatives). Notice that the high degree of supersymmetry (16 or 32 generators) completely determines the low energy action (2.1)[4].

This action is written in the so-called string frame: compared to the usual Einstein-Hilbert action or the action for a scalar field, there is a unusual factor $\mathrm{e}^{-2\Phi}$ and a wrong sign of the scalar field kinetic term. This can be changed by going to a physical frame called Einstein frame (labelled by an e) by a conformal transformation

$$G^e_{MN} = \Omega^2 G_{MN} = \mathrm{e}^{-\frac{4\Phi}{D-2}}G_{MN}. \tag{2.2}$$

Here $D = 10$, thus in the action (2.1) we have to replace G_{MN} by $\mathrm{e}^{+\Phi/2}G^e_{MN}$ which yields, for example

$$H_{ABC}H^{ABC} = H^e_{ABC}H_e^{ABC}\mathrm{e}^{-3\Phi/2}, \tag{2.3}$$

where the sub- or superscript e simply means that now the indices are pulled up and down with the metric in the Einstein frame. Under the conformal transfor-

[1] In the article in Chap. 5 we use the more compact notation $\int \mathrm{d}^{10}x\sqrt{-G}\frac{1}{5!}F_{ABCDE}F^{ABCDE} = \int F\wedge *F$ for the second integral, where $*F$ is the Hodge dual of F.

[2] Physically, the dilaton measures the size of the eleventh compactified dimension.

[3] It is well known that a 1-form, *e.g.* the electromagnetic potential A_μ, couples to the charge of a point particle which is 0-brane. In analogy, a 2-form potential couples to a 1-brane or a string, a 3-form potential couples to a 2-brane and so on.

[4] In string theory the action Eq. (2.1) corresponds to the low energy action of type IIB, and the fields in the first line, G_{MN}, H_{ABC}, and Φ, are associated with the Neuveu-Schwarz (NS-NS) sector, whereas C_{MNRS} with the Ramond-Ramond (RR-RR) sector.

mation (2.2) the Riemann scalar transforms as (see Ref. [160])

$$
\begin{aligned}
R &= \Omega^2 \left[R_e + 2(D-1)G_e^{MN}\nabla_M\nabla_N \ln\Omega - (D-2)(D-1)G_e^{MN}(\nabla_M \ln\Omega)(\nabla_N \ln\Omega)\right] \\
&= \mathrm{e}^{-\Phi/2}\left[R_e - \frac{9}{2}G_e^{MN}\nabla_M\nabla_N\Phi - \frac{9}{2}G_e^{MN}(\nabla_M\Phi)(\nabla_N\Phi)\right],
\end{aligned}
\tag{2.4}
$$

where R and R_e are the Riemann scalars constructed from the metrics G_{MN} and G^e_{MN} respectively. The action written in the Einstein frame is

$$
\begin{aligned}
S_{10}^e = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G_e}\Big(R_e &- \frac{1}{2}(\nabla_M\Phi)(\nabla^M\Phi) - \frac{1}{2}\frac{1}{3!}\mathrm{e}^{-\Phi}H^e_{ABC}H_e^{ABC} \\
&- \frac{1}{2}\frac{1}{5!}F^e_{ABCDE}F_e^{ABCDE}\Big),
\end{aligned}
\tag{2.5}
$$

where the exponential is not present anymore, and the dilaton kinetic term comes with the right sign and prefactor. We have used the fact that $\nabla_M\nabla^M\Phi = 0$ which is the dilaton equation of motion if the source is a 5-form (see below).

This action is of great importance for brane cosmology, for it admits solutions which are extended objects, so-called p-branes (see Chap. 3). Cosmologists often tend to use simplified effective theories in space-times with other dimensionalities than ten, mostly four, five or six, and to keep only fields which they can make use of. Therefore let us generalize the action to a D-dimensional space-time, the field strength to a q-form, and omit the field H_3. We shall also suppress the sub- or superscripts e as now all quantities are in the Einstein frame

$$
S_D = \frac{1}{2\kappa_D^2}\int \mathrm{d}^D x\sqrt{-G}\left(R - \frac{1}{2}(\nabla_M\Phi)(\nabla^M\Phi) - \frac{1}{2}\frac{1}{q!}\mathrm{e}^{a_q\Phi}F_{A_1\cdots A_q}F^{A_1\cdots A_q}\right),
\tag{2.6}
$$

which leads to the equations of motion

$$
\begin{aligned}
R_{MN} &= \frac{1}{2}(\nabla_M\Phi)(\nabla_N\Phi) \\
&\quad + \frac{1}{2}\frac{1}{(q-1)!}\mathrm{e}^{a_q\Phi}\left(F_{MA_2\cdots A_q}F_N{}^{A_2\cdots A_q} - \frac{q-1}{q(D-2)}F_{A_1\cdots A_q}F^{A_1\cdots A_q}G_{MN}\right), \\
0 &= \nabla_M\left(\mathrm{e}^{a_q\Phi}F^{MA_2\cdots A_q}\right), \\
\nabla_M\nabla^M\Phi &= \frac{1}{2}\frac{a_q}{q!}\mathrm{e}^{a_q\Phi}F_{A_1\cdots A_q}F^{A_1\cdots A_q},
\end{aligned}
\tag{2.7}
$$

where $a_q = (5-q)/2$.

2.3 String theory

2.3.1 Introductory remarks

Despite its great success, the $SU(3)\times SU(2)\times U(1)$ standard model, together with general relativity, is certainly not a complete description of nature. Firstly, the standard model depends on roughly twenty free parameters and is therefore too arbitrary. Secondly, attempts to unify quantum field theory with general relativity following the usual perturbative methods have failed, because the theory turned out to be non renormalizable. This indicates that at 'high energies' new elements come into play. And thirdly, the singularities of general relativity survive even in this unified theory.

A number of ways have been suggested to solve these problems. One idea is extra-dimensions, first suggested to unify electromagnetism and gravity. Another idea is to include supersymmetry, which resulted in supergravity theories. However, none of these concepts have led to a theory that solves all of the problems mentioned above. The first plausible and promising candidate for a working theory of quantum gravity is string theory.

From quantum field theory we are used to point-like particles that are excitations of some field and which interact locally by exchange of gauge particles. In string theory the fundamental constituents are strings, *i.e.* 1-dimensional objects. Like a chord, a string can oscillate, and there is a spectrum of masses or energies associated with the different oscillatory states. To a low energy observer an oscillating string looks like a point-like particle with a rest mass equal to the energy of the oscillatory state. In particular, a single string can give rise to different types of particles depending on its state of oscillation.

When building a consistent quantum theory based on strings one finds that: a) such a theory includes gravity: the spectrum contains a massless spin-2 state, the graviton, which arises as an excitation of a closed string. Moreover, the Einstein-Hilbert action of general relativity turns out to be part of the low energy action of string theory. This is the 'string theory miracle'. b) String theory is a consistent and divergence-free theory of quantum gravity. In particular, the ultraviolet divergence, arising from short distances, is absent. Roughly speaking this is due to the fact that extended objects such as strings 'smear out' the interactions over space-time, and this softens the UV divergence. The technical argument is that, when calculating string amplitudes, the UV divergent region is not in the domain of integration. c) String theory gauge groups, *e.g.* E_8, contain the standard model groups. d) A consistent string theory must be supersymmetric (called superstring theory), because bosonic string theory has a tachyon. Unlike in quantum field theory, fermions are now required for consistency. e) String theory has only one free parameter, $\alpha' = 1/2\pi\tau_F$, where τ_F is the string tension[5]. This sets the string length scale $\ell_s = \alpha'^{1/2}$ and the string mass scale $m_s = \alpha'^{-1/2}$. f) Superstring theory predicts the number of space-time dimensions to be $D = 10$. It requires the existence of extra-dimensions for consistency. Points a) and f) are particularly interesting for cosmology. If string theory really is the correct description of nature, one must ask the question how cosmology looks like within this framework.

There exist five types of superstring string theories, called type I, IIA, IIB,

[5]The subscript F indicates that it is a fundamental sting, in contrast to D-strings (1-branes) that we shall encounter later on, and to cosmic strings.

heterotic $SO(32)$, and heterotic $E_8 \times E_8$. Type I is a theory of open and closed unoriented superstrings, whereas type II contains only closed oriented superstrings. Here, I and II is the number of supersymmetry generators, and A and B indicate that the left- and right-moving oscillators transform under separate space-time supersymmetries that have opposite (A) or equal (B) chirality. These five string theories can be viewed being different corners in the moduli space of a single theory, called M theory[6]. A particular compactification of M theory due to Hořava and Witten is highly relevant for brane world models. We shall discuss it in paragraph 2.5.3.

It would be beyond the scope of this thesis (and the knowledge of its author) to enter into more details here. In the following sections, however, we shall discuss in some detail the string theory prediction of the space-time dimensionality, as well as the mass spectrum of a closed string, since these issues are of direct relevance for the articles in this thesis.

2.3.2 10- and 26-dimensional space-times

In general relativity, space-time is a given fundamental quantity, whose dynamics are described by Einstein's equations. The number of space-time dimensions is set to four by hand. In string theory space-time is a derived concept: the fundamental quantity is the world-sheet $\mathcal{W}$, swept out by a moving string, and the dynamics are described by the Polyakov action

$$S = -\frac{1}{4\pi\alpha'} \int_{\mathcal{W}} \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-g}\, g^{\mu\nu} (\partial_\mu X^M)(\partial_\nu X^N) G_{MN}. \tag{2.8}$$

Here, τ and σ are internal coordinates on the world-sheet with $-\infty < \tau < \infty$, $0 \leq \sigma \leq s$, where s is the length of a string, and $g_{\mu\nu}$ (with $\mu = \tau, \sigma$) is the internal metric. Diffeomorphism invariance on $\mathcal{W}$ requires that $X'^M(\tau', \sigma') = X^M(\tau, \sigma)$. Therefore, from the world-sheet perspective, the set $\{X^M\}$ corresponds to D massless scalar fields covariantly coupled by the metric G_{MN} (setting $M = 0, \cdots, D-1$). On $\mathcal{W}$ everything can be described in terms of a 2-dimensional conformal field theory. On the other hand, one may want to embed the world-sheet into a D-dimensional target space-time $\mathcal{M}^D$. Then, $X^M(\tau, \sigma)$ play the role of embedding functions. A priori we do not know, what properties $\mathcal{M}^D$ should have for the theory to be consistent, so for the moment we must leave the value of D open. Now here is the crucial point: specifying the 'couplings' of the D scalar fields to each other is equivalent to specifying G_{MN}. Therefore, space-time is encoded in a 2-dimensional conformal field theory, and string theory is able to predict its dimensionality. It is $D = 26$ for bosonic strings, and $D = 10$ for super strings. Since there is no intuitive explanation of this result, and since it is crucial for brane cosmology, we would like to present a short calculation following

[6] Much about M theory is yet obscure, and also concerning its name there are only guesses whether 'M' stands for matrix, membrane, mother, mystery, magic, or is an inversed 'W' for Witten.

Ref. [124] to assert this prediction. For simplicity, the demonstration is made for open bosonic strings.

We use light cone coordinates[7] $x^M = (x^+, x^-, x^i)$ where $x^\pm = (x^0 \pm x^1)/\sqrt{2}$, $i = 2, \cdots, D-1$, and the canonical momenta are denoted $p^M = (p^+, p^-, p^i)$. The aim is to determine D. The position of a point on the string is given by $x^M = X^M(\tau, \sigma)$, and the transverse embedding functions X^i satisfy the wave equation[8]

$$\partial_\tau^2 X^i = c^2 \partial_\sigma^2 X^i, \tag{2.9}$$

with velocity $c = s/2\pi\alpha' p^+$. We impose Neumann boundary conditions by requiring that the derivative tangential to the string vanishes ('no momentum is flowing off the string'): $\partial_\sigma X^i = 0$ at $\sigma = 0, s$. Then the general solution to Eq. (2.9) is[9]

$$X^i(\tau, \sigma) = x^i + \frac{p^i}{p^+}\tau + i(2\alpha')^{1/2} \sum_{n \neq 0} \frac{\alpha_n^i}{n} \mathrm{e}^{-i\pi n c\tau/s} \cos\left(\frac{\pi n \sigma}{s}\right), \tag{2.10}$$

where the sum runs over all n from minus infinity to plus infinity, except for $n = 0$. In fact, the $n = 0$ term corresponds to the center of mass momentum of the string[10]

$$p^i(\tau) = \frac{p^+}{s} \int_0^s \mathrm{d}\sigma\, \partial_\tau X^i(\tau, \sigma), \tag{2.11}$$

whereas the first term is center of mass position

$$x^i(\tau) = \frac{1}{s} \int_0^s \mathrm{d}\sigma\, X^i(\tau, \sigma). \tag{2.12}$$

Each oscillator mode n has an amplitude α_n^i which, upon quantization, becomes an operator satisfying the bosonic commutation relations

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m,-n} \tag{2.13}$$

for each i, j, n, m separately. The requirement that X^i be real leads to $(\alpha_n^i)^\dagger = \alpha_{-n}^i$. The Hamiltonian is [124]

$$H = \frac{1}{2p^+} \sum_{i=2}^{D-1} p^i p^i + \frac{1}{4p^+\alpha'} \left(\sum_{n=1}^{\infty} \sum_{i=2}^{D-1} \alpha_{-n}^i \alpha_n^i + \alpha_n^i \alpha_{-n}^i \right). \tag{2.14}$$

Note that creation and annihilation operators are not yet normal ordered. By using the commutation relations (2.13) for $m = n$ and summing over the $D-2$

[7] By using light cone coordinates, the diffeomorphism and Weyl redundancy of the Polyakov action is eliminated.

[8] After a particular gauge fixing of the induced metric on the world-sheet.

[9] This solution extends to X^+ and X^-, but for our purpose only the transverse coordinates are relevant.

[10] The exact definition of p^+ is made via the Lagrangian of the string, see Ref. [124]. Here, we content ourselves with the observation that p^+ corresponds to the energy of the string.

transverse coordinates, one finds

$$\sum_{i=2}^{D-1} \alpha^i_n \alpha^i_{-n} = \sum_{i=2}^{D-1} \alpha^i_{-n} \alpha^i_n + n(D-2). \tag{2.15}$$

Upon inserting this into Eq. (2.14) one encounters an infinite sum over all n which can be regularized to give

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}. \tag{2.16}$$

Thus the Hamiltonian becomes

$$H = \frac{1}{2p^+} \sum_{i=2}^{D-1} p^i p^i + \frac{1}{2p^+\alpha'} \left(\sum_{n=1}^{\infty} \sum_{i=2}^{D-1} \alpha^i_{-n} \alpha^i_n + \frac{2-D}{24} \right). \tag{2.17}$$

The mass squared of an oscillatory state is now

$$M^2 = 2p^+ H - \sum_{i=2}^{D-1} p^i p^i = \frac{1}{\alpha'} \left(N + \frac{2-D}{24} \right), \tag{2.18}$$

where we have replaced the sum over the oscillators by the level number N. Let us look at the first excited state with $N = 1$

$$\alpha^i_{-1}|0\rangle, \quad M^2 = \frac{1}{\alpha'} \frac{26-D}{24}. \tag{2.19}$$

Since i runs form 2 to $D-1$, these are $D-2$ internal states. This is exactly the amount of states that a *massless* spin 1 particle has in a D-dimensional space-time. This comes as follows: for a massless particle the momentum can be written as $p^M = (E, E, 0, \cdots, 0)$, hence the invariance group is the little group $SO(D-2)$. Now the spin 1 representation of $SO(D-2)$ is $(D-2)$-dimensional, giving rise to $D-2$ internal states. Therefore the state (2.19) must be massless which requires $D = 26$. Summarizing, by simply demanding Lorentz invariance, we have found that the bosonic string can only live in a target space-time that has 1 time and 25 spatial dimensions. A similar argument, including fermionic fields, shows that for superstrings the number of space-time dimensions must be $D = 10$.

These predictions, which appear to be wrong at first sight, are usually reconciled with the observed $D = 4$ by compactifying the superfluous dimensions (see Sec. 2.5). But here is the problem: the effective 4-dimensional theory, and consequently the physical constants in our universe, crucially depend on the details of the compactification process. Therefore, string theory looses virtually all of is predictive power. It is not clear how to overcome this deficiency. In Chap. 10 we discuss a dynamical 'decompactification' mechanism which possibly helps avoiding the problem.

2.3.3 Mass spectrum of a closed string on a compact direction

In preparation for part III of this thesis, we derive the mass spectrum of a closed string put on a compact direction. We shall encounter momentum, winding, and oscillatory modes which are interesting for string and brane gas cosmology.

A closed string solution of the wave equation (2.9) has to satisfy $X^M(\tau,\sigma+s) = X^M(\tau,\sigma)$, in addition to the Neumann boundary condition. The mode expansion is now

$$X^M(\tau,\sigma) = x^M + \frac{p^M}{p^+}\tau + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n\neq 0}\left(\frac{\alpha_n^M}{n}\mathrm{e}^{-2\pi n(ic\tau+i\sigma)/s} + \frac{\tilde{\alpha}_n^M}{n}\mathrm{e}^{-2\pi n(ic\tau-i\sigma)/s}\right). \tag{2.20}$$

We have replaced the index i by M, as all D space-time coordinates solve Eq. (2.9). It is easy to see that Eq. (2.20) is the correct mode expansion for the closed string by comparing it with the form (3.68) of the open string solution[11]. In both cases the first term of the oscillatory part corresponds to a left moving wave and the second to a right moving wave. On a closed string, those modes travel independently in opposite senses. Therefore, the two amplitudes α_n^M and $\tilde{\alpha}_n^M$ are different, whereas for the open string they are tied together by the Neumann boundary condition. Notice also that there is a factor 2π instead of π in the exponent to satisfy the periodicity condition mentioned above.

One may rewrite Eq. (2.20) in terms of complex coordinates $z = \mathrm{e}^{\sigma_0 + i\sigma}$ and $\bar{z} = \mathrm{e}^{\sigma_0 - i\sigma}$, where $\sigma_0 = ic\tau$ becomes real after an analytic continuation to a Minkowskian world-sheet. In the momentum term, τ can be expressed as $\tau = \ln(|z|^2)/(2ic)$. The length of the closed string is set to be $s = 2\pi$. Then the velocity defined under Eq. 2.9 becomes $c = 1/(\alpha' p^+)$. With these substitutions Eq. (2.20) reads

$$\begin{aligned}X^M(z,\bar{z}) &= X^M(z) + X^M(\bar{z})\\ &= \left(\frac{x^M}{2} - i\frac{\alpha'}{2}p_L^M\ln(z) + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n\neq 0}\frac{\alpha_n^M}{n}z^{-n}\right)\\ &\quad + \left(\frac{x^M}{2} - i\frac{\alpha'}{2}p_R^M\ln(\bar{z}) + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^M}{n}\bar{z}^{-n}\right).\end{aligned} \tag{2.21}$$

We have split the center of mass momentum p^M into left and right moving momenta p_L^M and p_R^M, in order to write the mode expansion as a sum of two independent parts: the 'left-movers' are described by a holomorphic function, the 'right'movers' by an anti-holomorphic function.

Let us focus for a moment on the zero modes. From Eq. (2.21) one finds, substituting back τ and σ,

$$X^M(\tau,\sigma) = x^M + \frac{\alpha'}{2}\left(p_L^M + p_R^M\right)c\tau + \frac{\alpha'}{2}\left(p_L^M - p_R^M\right)\sigma. \tag{2.22}$$

[11]which is just a rewriting of the open string mode expansion given in Eq. (2.10)

In a non compact space-time the embedding functions are single valued and, as one runs around the closed string, $X^M(\tau,\sigma+2\pi) = X^M(\tau,\sigma)$. Thus $p_L^M = p_R^M$, and Eq. (2.22) is just a trivial rewriting of Eq. (2.20). Suppose now that one direction, x^d say, is a circle of radius R. Then

$$X^d(\tau,\sigma+2\pi) = X^d(\tau,\sigma) + 2\pi R\omega, \tag{2.23}$$

where $2\pi R$ is the periodicity, and $\omega \in \mathbb{Z}$. This leads to the constraint

$$\frac{\alpha'}{2}\left(p_L^d - p_R^d\right) = \omega R. \tag{2.24}$$

On the other hand, the momentum flowing around a compact direction must be quantized according to[12]

$$p_L^d + p_R^d = \frac{n_L}{R} + \frac{n_R}{R} = \frac{2n}{R}, \quad n_L, n_R, n \in \mathbb{Z}. \tag{2.25}$$

From Eqs. (2.24) and (2.25) one finds

$$p_L^d = \frac{n}{R} + \frac{\omega R}{\alpha'}, \quad p_R^d = \frac{n}{R} - \frac{\omega R}{\alpha'}. \tag{2.26}$$

Momentum along a compact direction looks like mass in the remaining $D-1$ dimensions (see Eq. (2.47) later on). Intuitively, the momenta in Eq. (2.26) give therefore rise to an 'averaged' mass

$$M^2 \simeq \frac{1}{2}\left[\left(p_L^d\right)^2 + \left(p_R^d\right)^2\right] = \left(\frac{n}{R}\right)^2 + \left(\frac{\omega R}{\alpha'}\right)^2. \tag{2.27}$$

More carefully, one requires that the Virasoro generators[13] L_0 and $\tilde{L}_0$ vanish

$$\begin{aligned} 0 = L_0 &= \frac{1}{2}\frac{\alpha'}{2}\left[\left(p_L^d\right)^2 + p^2\right] + N - 1, \\ 0 = \tilde{L}_0 &= \frac{1}{2}\frac{\alpha'}{2}\left[\left(p_R^d\right)^2 + p^2\right] + \tilde{N} - 1, \end{aligned} \tag{2.28}$$

where p^2 denotes the space-time momentum in the $D-1$ non compact dimensions. N and $\tilde{N}$ are the total level number of left- and right-movers, and -1 is a constant from normal ordering the expressions. The $D-1$-dimensional mass is given by $M^2 = -p^2$. Inserting expressions (2.26) into (2.28) and solving for M^2 yields

$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{\omega R}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2). \tag{2.29}$$

[12]This can be seen as follows: the general translation operator is $\exp(ix^d p^d)$. Under a full translation around the compact direction, *i.e.* $x^d = 2\pi R$, a state must be invariant. Thus $\exp(i2\pi R p^d) = 1$, and hence $p^d = n/R$, where $n \in \mathbb{Z}$.

[13]A profound explication would be beyond the scope of this thesis. Here, we just remark that the Virasoro generators are coefficients of the Fourier expansion of the energy-momentum tensor on the world-sheet of a string. The requirement that $L_0 = 0$ and $\tilde{L}_0 = 0$ for left- and right-movers is a physical state or on-shell condition.

Eq. (2.29) is the main result of this section. It shows how momentum-, winding-, and oscillatory modes contribute the mass of a closed string. Notice that the contribution of the oscillators is set by the string tension which is proportional to $1/\alpha'$. The number ω is a winding number, counting how many times the string winds around the compact direction. It is positive or negative according to the orientation of the string. Two states with opposite winding $\pm\omega$ may annihilate, leaving a state with $\omega = 0$. Strings with non zero winding number are topological solitons. The existence of winding states is an intrinsically stringy phenomenon.

Furthermore, it can be shown that the combination $L_0 - \tilde{L}_0$ generates translations along the string. Hence, one requires additionally that $L_0 - \tilde{L}_0 = 0$ to ensures that the physics remain invariant. Subtracting the two equations in (2.28) leads to the 'level matching condition'

$$n\omega + N - \tilde{N} = 0. \tag{2.30}$$

If there are equally many left and right moving oscillators, either the center of mass momentum along the string or the winding number have to be zero. A possibility consistent with the level matching condition (2.30) is to set $n = \omega = N = \tilde{N} = 0$. Then the mass formula (2.29) gives

$$M^2 = -\frac{4}{\alpha'}, \tag{2.31}$$

which is precisely the mass of the tachyon of bosonic string theory. In superstring theory this mode is not present anymore.

We have presented the derivation of the closed string spectrum in some detail, because the result (2.29) will be crucial in string and brane gas cosmology.

2.4 Geometrical remarks on compact spaces

In the previous sections we have introduced and motivated the idea of extra-dimensions. Naive observation, however, suggests that we live in a universe with three spatial dimensions. How can the string theoretical prediction be reconciled with this observational fact? A simple explanation for why extra-dimensions (assuming that they exist) are invisible in our low energy world is the following: suppose that all n extra-dimensions are not 'large' and 'straight', but curled up to small circles of radius R. Then, they are visible only at energies above $\sim 1/R$. For instance, if R is the Planck length, it would take 10^{19} GeV to excite states associated with the extra-dimensions (see paragraph 2.5.1 for the precise meaning of this). At energies lower that that, our world looks 3-dimensional. Some properties of such 'Kaluza-Klein states' are described in Sec. 2.5, and more quantitative statements on the size of extra-dimensions are made in Sec. 2.7.

The aim of the present section is to become familiar with some geometrical properties of compact and non compact spaces. With a few 'reasonable' assumptions it is possible to exclude certain topologies for our universe as well as for the extra-dimensional space right from the start. We closely follow the treatment of Ref. [147].

2.4.1 Spaces with positive, negative, and zero Ricci curvature

Consider a gravity theory on a D-dimensional manifold $\mathcal{M}^D$ with the metric G and the action

$$S_D = \frac{1}{2\kappa_D^2}\int \mathrm{d}V R(G) + \text{matter}, \tag{2.32}$$

where $\mathrm{d}V$ is the invariant volume form on $\mathcal{M}^D$. By choosing local coordinates $x^M = (x^0, \cdots, x^{D-1})$, one can write $\mathrm{d}V = \mathrm{d}^D x\sqrt{-\det(G)}$. The constant κ_D^2 is related to the D-dimensional Newton constant, and $R(G)$ is the Riemann scalar constructed from the metric G.

We are looking for solutions which are locally of the form $\mathcal{M}^D = \mathcal{M}_1 \times \mathcal{M}_2$ and $G = \pi_1^* g_1 + \pi_2^* g_2$ under the map $\pi_i = \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_i$. In the following we take $(\mathcal{M}_1, g_1)$ to be a Lorentzian manifold and $(\mathcal{M}_2, g_2)$ a Riemannian manifold. Our universe is identified with $\mathcal{M}_1$ (when $\dim(\mathcal{M}_1) = 4$), or with a subspace of $\mathcal{M}_1$. The manifold $\mathcal{M}_2$ contains the remaining extra-dimensions. Both are assumed to be Einstein spaces, *i.e.*

$$Ric(g_1) = C_1 g_1, \quad Ric(g_2) = C_2 g_2. \tag{2.33}$$

In component language, the Riemann tensor of a n-dimensional Einstein space reads

$$R_{ABCD} = \frac{1}{n(n-1)} R(g_{AC}g_{BD} - g_{AD}g_{BC}), \tag{2.34}$$

and the Ricci tensor is

$$R_{AB} = g^{MN} R_{MANB} = \frac{1}{n} g_{AB} R. \tag{2.35}$$

The normalization in Eq. (2.34) is chosen such that the contraction $g^{AB}R_{AB}$ indeed yields the Riemann scalar R. Finally, the Einstein tensor is

$$\mathcal{G}_{AB} = R_{AB} - \frac{1}{2} g_{AB} R = \frac{2-n}{2n} g_{AB} R. \tag{2.36}$$

Eqs. (2.34)-(2.36) are valid for $\mathcal{M}_1$ or $\mathcal{M}_2$ with $n = \dim(\mathcal{M}_i)$.

The simplest Lorentzian manifolds $(\mathcal{M}_1, g_1)$ satisfying the relations (2.33) are those with maximal symmetry, namely de Sitter ($C_1 > 0$), Minkowski ($C_1 = 0$), and anti-de Sitter ($C_1 < 0$). If $\mathcal{M}_1$ is embedded in a $\tilde{D}$-dimensional flat space with $\tilde{D} = \dim(\mathcal{M}_1) + 1$, then the isometry groups of de Sitter and anti-de Sitter space are $SO(1, \tilde{D}-1)$ and $SO(2, \tilde{D}-2)$, respectively (see *e.g.* Eq. (3.66)). To proceed we make the 'reasonable' assumption that $C_1 \leq 0$ in order to satisfy the strong energy condition (1.33) on $\mathcal{M}_1$.

For $(\mathcal{M}_2, g_2)$ we could for example take a sphere ($C_2 > 0$) or a torus ($C_2 = 0$). When $\mathcal{M}_2$ has positive Ricci curvature ($C_2 > 0$), one can prove the following theorem:

Theorem 1. *(Myers): Let $(\mathcal{M}_2, g_2)$ be an n-dimensional connected complete Riemannian manifold with positive definite Ricci curvature $Ric \geq (n-1)k_0$. Then*
(i) *The diameter of $\mathcal{M}_2$ is at most $\pi/\sqrt{k_0}$.*
(ii) *$\mathcal{M}_2$ is compact.*

The first point is relevant, for example, in the proposal of 'large' extra-dimensions in Sec. 2.7: the size of the extra-dimensions cannot be chosen independently of their geometry. The second point tells us, that the extra-dimensions are compact if $C_2 > 0$ (if $\mathcal{M}_2$ is simply connected). As mentioned at the beginning, the assumption of compact extra-dimensions is commonly made as a way to render them 'invisible'. For a proof of this theorem, see *e.g.* Ref. [143].

When $\mathcal{M}_2$ has negative Ricci curvature, then the following theorem applies (see Ref. [95]):

Theorem 2. *(Bochner): Let $(\mathcal{M}_2, g_2)$ be a compact Riemannian manifold with negative definite Ricci curvature. Then, there exist no non trivial Killing fields. In other words, if X is a Killing field, then $X = 0$.*

And finally, if $\mathcal{M}_2$ has zero Ricci curvature (see Ref. [95]):

Theorem 3. *If $(\mathcal{M}_2, g_2)$ is a compact Riemannian manifold with vanishing Ricci tensor field, then every infinitesimal isometry of $\mathcal{M}_2$ is a parallel vector field. In other words, any Killing field X is parallel transported, $\nabla X = 0$ (where ∇ denotes the connection on $\mathcal{M}_2$).*

From this, one derives the

Corollary 1. *(Lichnerowicz, Ref. [95]): If a connected compact homogeneous Riemannian manifold $(\mathcal{M}_2, g_2)$ has zero Ricci tensor field, then $\mathcal{M}_2$ is a Euclidean torus.*

We shall often refer to this case later on, for instance, when we discuss toroidal compactification and Kaluza-Klein theories. In the brane gas model, presented in part III of this thesis, all spatial dimensions are assumed to have the topology of a torus. Unfortunately, this geometry does not lead to phenomenologically interesting particle physics models, since the isometry group of the manifold is the gauge group, which in the case of a torus turns out to be abelian. More realistic compactifications involve Calabi-Yau spaces.

2.4.2 The gravitational field equations

In the previous paragraph, we considered the case $C_1 \leq 0, C_2 > 0$ as potentially interesting for brane world models. Our universe would be contained in $\mathcal{M}_1$ and the space of the remaining extra-dimensions is $\mathcal{M}_2$. Let us now take a look at the corresponding field equations. With the assumptions $\mathcal{M}^D = \mathcal{M}_1 \times \mathcal{M}_2$ and $G = \pi_1^* g_1 + \pi_2^* g_2$, the connection forms and the curvature forms separate, and hence the Ricci tensor of $\mathcal{M}^D$ splits into

$$Ric(G) = \pi_1^* Ric(g_1) + \pi_2^* Ric(g_2). \tag{2.37}$$

Notice, that this is not the case for the Einstein tensor. In a pure gravity theory, *i.e.* with no cosmological constant, the field equation is

$$Ric(G) = 0, \tag{2.38}$$

which forces $C_1 = C_2 = 0$. $C_1 = 0$ is desirable, as it represents a Minkowski vacuum, but $C_2 = 0$ leads to uninteresting gauge groups. We therefore include a cosmological constant Λ_D into the Einstein equation,

$$\mathcal{G}(G) + \Lambda_D G = 0. \tag{2.39}$$

The Ricci tensor now reads

$$Ric(G) = \frac{2\Lambda_D}{D-2} G, \tag{2.40}$$

and the Riemann scalar is

$$R(G) = \frac{2D}{D-2} \Lambda_D. \tag{2.41}$$

The difference between C_1 and C_2 is set by Λ_D. Particularly interesting for brane world cosmology is the case $C_1 < 0, C_2 > 0$, where $\mathcal{M}_1$ is a 5-dimensional anti-de Sitter space-time and $\mathcal{M}_2$ a 5-sphere. The space-time $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ is then $\mathrm{AdS}_5 \times \mathrm{S}^5$. Eq. (2.41) is valid also in the subspace $\mathcal{M}_1$, such that the cosmological constant of AdS_5 is given by

$$\Lambda = \frac{n-2}{2n} R(g_1) \tag{2.42}$$

with $n = 5$. Since the Ricci curvature of AdS_5 is negative, also the 5-dimensional cosmological constant is negative[14]. We shall derive the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry in Sec. 3.3, and show that our universe could be a 3-brane embedded in AdS_5 .

We end this section with the following observation. Identify our 4-dimensional universe with $\mathcal{M}_1$. For the extra-dimensions to be compact, chose again $C_2 > 0$, and to have them of Planckian size, $C_2 \sim M_4^2$, according to Myers' theorem. Here, M_4 denotes the 4-dimensional Planck mass. One 'naturally' expects that $C_1 \sim C_2$, such that the cosmological constant in $\mathcal{M}_1$ is of the order M_4^2, and thus 120 orders of magnitude too large.

[14]In Chap. 6 we are concerned with cosmological models that are linked to the AdS_5 geometry by a coordinate transformation. Therefore, in those models the bulk cosmological constant is also negative.

2.5 Toroidal compactification

As a particular example we discuss toroidal compactification. It serves to illustrate some features which are common also to other compactifications, and it will lead us to the Kaluza-Klein unification of gauge theories and gravity. Consider a D-dimensional manifold with a metric, $(\mathcal{M}^D, G)$, which can be decomposed locally into

$$\mathcal{M}^D = \mathcal{M}^d \times S^1, \quad x^M = (x^\mu, x^d). \tag{2.43}$$

Here, S^1 is a circle with radius R, and x^d a periodic coordinate, *i.e.*

$$x^d = x^d + 2\pi R. \tag{2.44}$$

This is an example of toroidal compactification. In general, further or even all spatial dimensions can be periodic.

2.5.1 Kaluza-Klein states

To see the effect of periodicity, consider a massless scalar field $\phi(x^M)$ in a D-dimensional Minkowski space-time. The x^d dependence can be expanded into a Fourier series

$$\phi(x^M) = \phi(x^\mu, x^d) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu) \mathrm{e}^{inx^d/R}. \tag{2.45}$$

The D-dimensional wave equation for a free scalar field, $\partial_N \partial^N \phi(x^M) = 0$, yields

$$\left(\partial_\mu \partial^\mu - \frac{n^2}{R^2}\right) \phi_n(x^\mu) = 0, \tag{2.46}$$

which is an equation of motion for the Fourier coefficients ϕ_n. The ϕ_n's are therefore d-dimensional scalar fields with masses

$$M_n^2 = \frac{n^2}{R^2}, \quad n^2 \geq 0. \tag{2.47}$$

Thus, the d-dimensional mass or energy spectrum is an infinite tower of equally spaced states, where the spacing is given by the size of the compactified dimension, see Fig. 2.1. The lowest state is massless, and it takes an energy $1/R$ to excite the first massive state. At energies much lower than this, only the massless ground state is relevant, and the physics are effectively d-dimensional. Hence, for any field in the theory, one can integrate out its dependence on the periodic direction to obtain a low energy effective description.

Now, consider a curved space-time where again the x^d direction is compact. The D-dimensional metric can be Fourier expanded according to

$$G_{MN}(x^M) = G_{MN}(x^\mu, x^d) = \sum_{n\in\mathbb{Z}} G^n_{MN}(x^\mu) \mathrm{e}^{inx^d/R}. \tag{2.48}$$

Assuming that the metric excitations are very massive, we can restrict ourselves to the zero mode, $G^0_{MN}(x^\mu)$. The issue now is to explain the Kaluza-Klein unification

Figure 2.1: Infinite tower of Kaluza-Klein mass states.

of the electromagnetic and gravitational fields. To that end, parameterize the zero mode (omitting the superscript) as

$$\mathrm{d}s^2 = G_{MN}\mathrm{d}x^M\mathrm{d}x^N = G_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \phi(\mathrm{d}x^d + A_\mu\mathrm{d}x^\mu)^2. \tag{2.49}$$

The fields $G_{\mu\nu}$, A_μ, and $\phi \equiv G_{dd}$ depend only on the non compact coordinates x^μ. In a more suggestive way, we write the decomposition (2.49) in matrix form

$$(G_{MN}) = \begin{pmatrix} G_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\mu & \phi \end{pmatrix}. \tag{2.50}$$

From the d-dimensional point of view, the D-dimensional metric separates into $g_{\mu\nu} \equiv G_{\mu\nu} + \phi A_\mu A_\nu$, A_μ, and ϕ, which transform as a tensor, vector, and scalar under the irreducible representations of some symmetry group in $\mathcal{M}^d$. The tensor $g_{\mu\nu}$ is the metric in the subspace $\mathcal{M}^d$, and the vector A_μ represents the electromagnetic potential. They are both components of a higher-dimensional, purely gravitational field. In the original formulation of Kaluza and Klein, the space-time $\mathcal{M}^D$ was taken to be 5-dimensional, in order to unify the observed 4-dimensional gravitational and electromagnetic fields. Clearly, the idea also holds if d is arbitrary.

Furthermore, note that Eq. (2.49) is the most general form of the metric invariant under reparametrizations

$$x^d \to x'^d = x^d + \lambda(x^\mu), \tag{2.51}$$

if simultaneously

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\lambda. \tag{2.52}$$

This is easily seen by applying the transformation laws:

$$\mathrm{d}x'^d + A'_\mu \mathrm{d}x^\mu = \mathrm{d}x^d + \mathrm{d}\lambda + A_\mu \mathrm{d}x^\mu - (\partial_\mu \lambda)\mathrm{d}x^\mu = \mathrm{d}x^d + A_\mu \mathrm{d}x^\mu . \tag{2.53}$$

Notice that the scalar λ does not depend on x^d, and thus $\mathrm{d}\lambda = (\partial_\mu \lambda)\mathrm{d}x^\mu$. Hence, remarkably, the d-dimensional gauge transformations (2.52) arise from higher-dimensional coordinate transformations. The conserved quantity associated with invariance under (2.51) is the momentum along S^1.

Unfortunately, the theory of Kaluza and Klein does not work in practice, because the field ϕ acts as a Brans-Dicke scalar, which modifies 4-dimensional gravity in a non acceptable way. Nevertheless, their idea shows the virtues (and dangers) of a higher-dimensional approach, and it is the prototype of other unifying theories.

Nowadays, string theory is the most promising candidate of such a theory. We have seen in Sec. 2.3, that it predicts a 10-dimensional space-time. To make contact with the observed world, the first step is to take the low energy limit, in which case the action of type IIB string theory is given by Eq. (2.1). The second step is to compactify six spatial dimensions. In the next paragraph, we show how this affects the low energy action.

2.5.2 Dimensional reduction of the action

Consider the action (2.1) and, for simplicity, keep only the Riemann scalar and the dilaton kinetic term. One then obtains the 'dilaton-gravity action'

$$S_D = \frac{1}{2\kappa_D^2} \int \mathrm{d}^D x \sqrt{-G}\, \mathrm{e}^{-2\Phi} \left(R + 4(\nabla_M \Phi)(\nabla^M \Phi) \right) . \tag{2.54}$$

Commonly, this action is used for $D = 10$, $D = 5$, and $D = 4$, passing from one to the other by a dimensional reduction. To see how this works in detail, assume that the x^d-direction is compact (where $D = d + 1$). The dependence on x^d can then be Fourier transformed according to Eq. (2.45), and only the zero modes are kept, for instance, $\Phi = \Phi(x^\mu)$, and the index on the derivative M is replaced by μ.

The Ricci scalar for the metric (2.49) can be decomposed as

$$R = R_d - 2\mathrm{e}^{-b}(\nabla_\mu \nabla^\mu \mathrm{e}^b) - \frac{1}{4}\,\mathrm{e}^{2b} F_{\mu\nu} F^{\mu\nu} , \tag{2.55}$$

where we have defined $\mathrm{e}^{2b} \equiv G_{dd}$, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, and R is constructed from the full D-dimensional metric G_{MN}, whereas R_d from $G_{\mu\nu}$ (but not from $g_{\mu\nu} \equiv G_{\mu\nu} + \mathrm{e}^{2b} A_\mu A_\nu$). Inserting this into the action (2.54) yields

$$\begin{aligned} S_d = \frac{1}{2\kappa_D^2} \int \mathrm{d}x^d \int \mathrm{d}^d x \sqrt{-G_d}\, \mathrm{e}^b \mathrm{e}^{-2\Phi} \Big(& R_d - 2\mathrm{e}^{-b}(\nabla_\mu \nabla^\mu \mathrm{e}^b) \\ & + 4(\nabla_\mu \Phi)(\nabla^\mu \Phi) - \frac{1}{4}\,\mathrm{e}^{2b} F_{\mu\nu} F^{\mu\nu} \Big), \end{aligned} \tag{2.56}$$

where G_d denotes the determinant of $G_{\mu\nu}$. The integral over x^d simply gives $2\pi R$, and the factor e^b, coming from the volume element of the compact direction, can be absorbed by defining a d-dimensional dilaton $\Phi_d(x^\mu) = \Phi(x^\mu) - b(x^\mu)/2$. This leads to

$$\begin{aligned} S_d = \frac{\pi R}{\kappa_D^2} \int \mathrm{d}^d x \sqrt{-G_d}\, \mathrm{e}^{-2\Phi_d} \Big(& R_d - (\nabla_\mu b)(\nabla^\mu b) \\ & + 4(\nabla_\mu \Phi_d)(\nabla^\mu \Phi_d) - \frac{1}{4}\, \mathrm{e}^{2b} F_{\mu\nu} F^{\mu\nu} \Big). \end{aligned} \tag{2.57}$$

We have assumed that the equation of motion for the modulus field b is $\nabla_\mu \nabla^\mu b = 0$, such that the corresponding term from the derivative is zero. The dimensional reduction leads to kinetic terms for the fields $b(x^\mu), \Phi_d(x^\mu), A_\mu(x^\mu)$. In Sec. 2.2, we have seen that the wrong sign and prefactor of the dilaton kinetic term is due to the fact that the action is written in the string frame instead of the physical Einstein frame. On the other hand, the dimensional reduction itself does not lead to potential terms, and so the fields $b(x^\mu), \Phi_d(x^\mu), A_\mu(x^\mu)$ are massless.

In addition to toroidal compactifications, there exist also compactifications on Calabi-Yau manifolds. These manifolds have a more complex structure and lead to phenomenologically more realistic particle physics theories.

2.5.3 Hořava-Witten compactification

For brane world cosmologies, a compactification scheme due to Hořava and Witten [77] is particularly interesting, and we shall briefly discuss it here. In this scenario, one starts from 11-dimensional M theory, which contains the five 10-dimensional superstring theories in different corners of its moduli space. Assume that the eleventh dimension is curled up to a circle S^1, and that this circle has a mirror symmetry Z_2. The coset space S^1/Z_2 is called an orbifold[15]. If θ denotes the coordinate on S^1, there are two fixed points at $\theta = 0$ and $\theta = \pi$, at which the coset space becomes singular. This can be avoided by placing a 10-dimensional hypersurface at each singular point. Since those hypersurfaces with nine spatial and one time direction have particular properties, *e.g.* a tension, they correspond to 9-branes[16]. As a condition for stability, the sum of the two tensions must be zero. In other words, there is a positive and a negative tension brane. In a cosmological framework, the sign of the tension is important, for it determines whether gravity on a brane is attractive or repulsive. Moreover, the positivity of the tension is a condition for local stability.

So far, space-time has the structure $\mathbb{R}^{10} \times S^1$. Now, on each 9-brane, one can compactify six spatial dimensions to end up with two parallel 3-branes residing at the orbifold fixed points. The resulting space-time is effectively 5-dimensional, and the S^1 plays the role of the fifth dimension.

[15] In general, an orbifold is the coset space $\mathcal{M}/H$, where H is a group of discrete symmetries of the manifold $\mathcal{M}$.

[16] The meaning and properties of p-branes are introduced in Chap. 3

This setup is very common in brane cosmology, where one of the two 3-branes is identified with our universe. Particularly interesting is that, before compactification, each of the 9-branes carries an E_8 gauge group of the heterotic string. After compactification, the E_8 is broken to $SU(3) \times SU(2) \times U(1)$, and hence the standard model of particle physics can be accommodated on a brane.

2.6 The modifications of Newton's law

Gravity propagates in all dimensions, because it is the dynamics of space-time itself. In particular, it is sensitive to the presence of extra-dimensions. This is not the case for the electromagnetic, the strong, and the weak forces. In string theory, the graviton corresponds to an excitation of a closed string, and as such it is free to propagate everywhere. Gauge particles, instead, correspond to open string ends, which are confined to lower-dimensional hypersurfaces (see Sec. 3.4).

In this section, we study the modifications of Newton's law in the presence of n extra-dimensions. In principle, this allows to detect extra-dimensions in a laboratory experiment.

2.6.1 Newton's law in d non compact spatial dimensions

First, let us generalize Newton's law to a number $d = 3 + n$ non compact spatial dimensions. Given a mass M, we want to find the force F exerted on a test mass μ a distance r away. In a d-dimensional space, the source M is enclosed by a $(d-1) = (2+n)$-dimensional sphere S^{2+n}. Gauss' theorem is

$$\int_{S^{2+n}} \mathbf{F} \cdot \mathbf{dS} = \int_{B^{3+n}} (\nabla \cdot \mathbf{F})\, \mathrm{d}V, \tag{2.58}$$

where B^{3+n} is a $3+n$-dimensional ball, whose surface is S^{2+n}. Since $\mathbf{F}$ is perpendicular onto every surface element $\mathbf{dS}$, the left hand side gives

$$F \int_{S^{2+n}} \mathrm{d}S = F r^{2+n} |S^{2+n}|, \tag{2.59}$$

where the surface area of the $(2+n)$-sphere is given by $|S^{2+n}| = \frac{2\pi^{(3+n)/2}}{\Gamma((3+n)/2)}$. To evaluate the right hand side of Eq. (2.58), we use the gravitational field equation[17]

$$\nabla \cdot \mathbf{g} = |S^{2+n}| G_D \rho, \tag{2.60}$$

where G_D denotes Newton's constant in $D = d + 1$-dimensional space-time. The normalization is chosen to cancel the surface area on the left hand side. Recall that for $n = 0$, this factor gives 4π. Substituting $\mathbf{F} = \mu \mathbf{g}$, we get

$$\mu \int_{B^{3+n}} (\nabla \cdot \mathbf{g})\, \mathrm{d}V = \mu |S^{2+n}| G_D \int_{B^{3+n}} \rho\, \mathrm{d}V = \mu |S^{2+n}| G_D M. \tag{2.61}$$

[17] The second field equation would be $\nabla \wedge \mathbf{g} = 0$, *i.e.* the gravitational field has no curl.

Equating Eqs. (2.59) and (2.61) gives

$$F = G_D \frac{\mu M}{r^{2+n}}. \tag{2.62}$$

This is Newton's law if there are n non compact extra-dimensions. The factor $1/r^n$ reflects the dilution of field lines into the additional space. Notice also the appearance of the D-dimensional instead of the 4-dimensional gravitational constant.

2.6.2 Newton's law with n compact extra-dimensions

Now assume that there are n compact extra-dimensions, each of size L, in addition to the usual three non compact directions. This situation can be reproduced by a setup with $3+n$ non compact dimensions and mirror masses M placed at the points kL, $k \in \mathbb{Z}$, along each of the extra-dimensions. Again, we want to find the force F on a test mass μ at a distance r from the origin. If $r \ll L$, the influence of the mirror masses is negligible, and the result is the same as for a single source M and $d = 3+n$ non compact spatial dimensions, derived in the previous paragraph:

$$F = G_D \frac{\mu M}{r^{2+n}} \quad \text{for} \quad r \ll L. \tag{2.63}$$

In particular, this is true if r is a distance in our universe. This allows us to test, in principle, whether there exist extra-dimensions by measuring the force between two test bodies in the laboratory. There is no need to access the extra-dimensions as the lines of force do it for us.

At distances $r \gg L$, a test mass feels the influence of the mirror masses, but it cannot discern their discrete spacing. Let us take r to be a distance in our universe, and discuss first the case $n = 1$. Then the mirror masses look like line of uniform mass density M/L. For symmetry reasons, we enclose it with a cylinder of length L_c and suitable end caps whose contribution will vanish in the limit $L_c \to \infty$, see Fig. 2.2. Again, we use Gauss' law (2.58), where on the left hand side the integration domain is now the surface of the cylinder $\partial C = \mathbb{R} \times S^2$. Then

$$F \int_{\partial C} \mathrm{d}S = F L_c (4\pi r^2). \tag{2.64}$$

For the right hand side of (2.58) one uses again the field equation (2.60) (with $n = 1$) as all dimensions are non compact

$$\mu |S^{2+1}| G_D \int_C \rho \,\mathrm{d}V = \mu |S^{2+1}| G_D \left(L_c \frac{M}{L} \right). \tag{2.65}$$

Solving for F one finds

$$F = \frac{|S^{2+1}| G_D}{4\pi L} \frac{\mu M}{r^2}. \tag{2.66}$$

The artificial cut-off L_c has dropped out. It is now straightforward to generalize

Figure 2.2: A uniform mass density (represented by the solid line) along the extra-dimension y is enclosed by a cylinder of length L_c with two hemispheres as endcaps. Here, r is a distance 'along' our 3-dimensional universe.

this result to n extra-dimensions by replacing the mass density by M/L^n, the integration domain by $\partial C = \mathbb{R} \times \cdots \times \mathbb{R} \times S^2$, and the surface area of the sphere in the field equations by $|S^{2+n}|$. The integral over ρ gives then $L_c^n(M/L^n)$. Summarizing,

$$F = \frac{|S^{2+n}|G_D}{4\pi L^n}\frac{\mu M}{r^2} \quad \text{for} \quad r \gg L. \tag{2.67}$$

In terms of the original description without mirror masses, we have found Newton's law for the gravitational attraction between two masses μ and M at a distance r in our 3-dimensional universe, but in the presence of n compact extra-dimensions. It is still a $1/r^2$ law, independently of n, as the curled up directions cannot be seen from distances large compared to L. Intuitively, since the extra-dimensional are compact, there is no room for the gravitational field lines to spread into them.

As a by-product of the result (2.67), one can read off the value of the 4-dimensional Newton constant (measured in gravitational strength experiments) in terms of the fundamental D-dimensional Newton constant.

$$G_4 = \frac{|S^{2+n}|G_D}{4\pi L^n}. \tag{2.68}$$

The main result of this section is that in the presence of n extra-dimensions, Newton's law is $F \sim 1/r^{2+n}$ at distances much smaller than the size of the extra-dimensions. Surprisingly, the $1/r^2$ form is experimentally confirmed only above

$r \simeq 20\,\mu$m [42]. From the experimenter's point of view, this is due to the weakness of gravity compared to the Coulomb force. It is extremely difficult to eliminate disturbing influences such as small electric charges on the probes to obtain a clear signal. The lower limit on the validity of Newton's law is at the same time the upper limit on the size of L. Extra-dimensions are thus not excluded as long as their size is smaller than $20\,\mu$m.

Whereas gravity is sensitive to extra-dimensions, gauge interactions are not. This is because gauge particles are confined to our universe (see Sec. 3.4). For that reason, extra-dimensions have not been detected in collider experiments, which test gauge interactions down to $1/(200\,\mathrm{GeV})$.

2.7 The hierarchy problem

The Einstein-Hilbert action of general relativity is

$$S_4 = \frac{1}{16\pi G_4}\int \mathrm{d}^4x\sqrt{-g}\,R(g) \equiv \frac{M_4^2}{2}\int \mathrm{d}^4x\sqrt{-g}\,R(g). \tag{2.69}$$

Here, we are interested only in the prefactor which has dimensions of a mass squared in order to make the action dimensionless. This mass is called the **Planck mass**[18] and is defined via the only free parameter in general relativity, Newton's constant

$$M_4 \equiv \frac{1}{\sqrt{8\pi G_4}} = 2.4\times 10^{18}\mathrm{GeV}. \tag{2.70}$$

This relation defines the 'energy scale of gravity'. For gauge interactions, instead, an important scale is the 'electroweak scale' defined as the energy at which the running coupling constants α_{EM} and α_W are of the same size, and the electromagnetic and the weak force are unified to a $SU(2)\times U(1)$ gauge theory. The numerical value of the electroweak scale is roughly 1 TeV.

Why do these two scales differ by a factor 10^{15}? Another way of formulating this 'hierarchy problem' is: why is gravity so much weaker than the the other fundamental forces? If one attempts to unify gravity and gauge interactions, their coupling constants should be at least of the same order, otherwise one has to specify a mechanism which explains this huge difference. In the following, we are discussing an idea to solve the hierarchy problem.

2.7.1 Fundamental and effective Planck mass

In Sec. 2.6 we have seen that the 4-dimensional Newton constant is related to the D-dimensional gravitational constant by Eq. (2.68). Here, we derive this relation in an alternative way, and in terms of mass scales rather than gravitational

[18]More precisely, Eq. (2.70) defines the *reduced* Planck mass in which numerical factors of order one and factors of π are absorbed.

constants. The action of a D-dimensional theory (supergravity, low energy string theory) is

$$S_D = \frac{1}{2\kappa_D^2}\int \mathrm{d}^D x\sqrt{-G}\,(R+\cdots) \equiv \frac{M_D^{D-2}}{2}\int \mathrm{d}^D x\sqrt{-G}\,(R+\cdots), \tag{2.71}$$

where we have defined a **D-dimensional Planck mass** M_D. The power on M_D is $D-2 = 2+n$ in order to make the action dimensionless. Note that, as we are interested in physical masses, we have placed ourselves in the Einstein frame.

Assuming that the x^d direction is a circle, and carrying out a dimensional reduction as described in Sec. 2.5, one finds

$$S_D = \underbrace{\frac{M_D^{D-2}}{2}\int \mathrm{d}x^d\sqrt{G_{dd}}}_{=\frac{M_{D-1}^{D-3}}{2}}\int \mathrm{d}^{D-1}x\sqrt{-G_d}\,(R_d+\cdots). \tag{2.72}$$

Therefore, the $(D-1)$-dimensional Planck mass is given by[19]

$$M_{D-1}^{D-3} = M_D^{D-2}V, \tag{2.73}$$

where V the volume of the compactified dimension. To arrive at an effective theory in four space-time dimensions, this procedure is repeated for the remaining $n-1$ extra-dimensions, and the overall coefficient is identified with the 4-dimensional Planck mass

$$M_4^2 = M_D^{D-2}V_n = M_D^{2+n}V_n, \tag{2.74}$$

where V_n denotes the volume of the n-dimensional compact space. Note that this formula holds not only for toroidal, but all kinds of compactifications in which space-time has a product structure $\mathcal{M}^D = \mathcal{M}_1 \times \mathcal{M}_2$, where our universe is a (sub)space of $\mathcal{M}_1$ and the space of extra-dimensions is $\mathcal{M}_2$.

2.7.2 The idea of 'large' extra-dimensions

A solution to the hierarchy problem based on the relation (2.74) was suggested by the authors of Refs. [12, 13]. Their idea is that the 4-dimensional Planck mass is only a derived scale, whereas the relevant scale is the D-dimensional Planck mass which could be around 1 TeV.

Let us write $V_n = L^n$ where L is the size of an extra-dimension. Setting $M_D \simeq 1\,\mathrm{TeV}$, Eq. (2.74) gives

$$L \simeq 10^{(30/n)-17}\mathrm{cm}. \tag{2.75}$$

[19]Note that in the Einstein frame the factor $\sqrt{G_{dd}}$ cannot be absorbed in a redefinition of the dilaton.

For $n = 1$, $L \simeq 10^{13}\,\mathrm{cm}$, which is excluded because in this case the modified Newton law (2.63) would apply up to solar-system distances. Obviously this is not the case. For $n = 2$, however, $L \simeq 1\,\mathrm{mm}$ which is only marginally excluded by experiments (see the end of Sec. 2.6). Therefore, a scenario with two 'large' extra-dimensions could possibly solve the hierarchy problem. To satisfy simultaneously other experimental constraints, coming for example from the production of massless higher-dimensional gravitons at the TeV scale, at least three extra-dimensions are actually needed. One also has to keep in mind that this result relies on the assumptions that the fundamental Planck mass really is around 1 TeV, and that the space-time does have a product structure. In Chap. 6 we present an alternative attempt to solve the hierarchy problem due to Randall and Sundrum.

Chapter 3

Branes

3.1 Extended objects

In general relativity the elementary objects are point-like particles moving on a four-dimensional manifold with a metric and a certain symmetry group. Those particles have to be test particles in the sense that their masses are negligible, otherwise the metric along the trajectory becomes singular. In quantum field theory, the elementary objects are approximated by point-like particles, although their wave function is spread in space.

The first example of classical elementary extended objects were topological defects formed during phase transition, *e.g.* cosmic strings and domain walls. In a 4-dimensional space-time domains walls are the biggest (apart from space-filling) objects that can exist. A D-dimensional space-time, however, can be populated by extended objects, having up to $p = D - 1$ spatial dimensions. A general definition of such p-branes is [37]: a p-brane is a dynamical system defined in terms of fields with support confined to a $(p+1)$-dimensional world-sheet surface in a background space-time manifold of dimension $D \geq p + 1$. The category p-brane includes particles (zero-branes), strings (1-branes), membranes (2-branes), and so on, up to space filling branes ($p + 1 = D$), which are continuous media, and where the confinement condition is redundant. The latter are not of primary interest for us. In this context, our 4-dimensional universe is conceived as a 3-brane, to which gauge fields and fermionic fields are confined. String theory provides a natural explanation for why matter can be trapped on a hypersurface. In addition, there exist also more phenomenological approaches [133].

We begin this chapter with some geometrical preliminaries, which are essential to describe p-branes. Section 3.3 explaines how p-branes emerge as solitonic solutions of supergravity, and how anti-de Sitter space arises in this context. In Sec. 3.4 we introduce Dp-branes from a string-theoretical point of view.

3.2 Differential geometrical preliminaries

This section serves as an introduction to a few geometrical concepts which are useful in brane cosmology. We try to give a self-consistent overview, mainly following Refs. [37] and [150]. The theory is fully covariant and applies to classical relativistic branes. These include supergravity p-branes, string theory D-branes in the low energy limit, as well as topological defects and brane gases. It also covers a wide range of physical systems such as metal plates in an electromagnetic field or soap bubbles. In all cases, it is assumed that the branes are infinitely thin, which is a good approximation if their curvature radius is much larger than their actual thickness.

In the first part, we define the first and second fundamental form for an arbitrary number of co-dimensions. Afterwards, we restrict ourselves to one co-dimension and discuss the equations of Gauss, Codazzi, and Mainardi, as well as the junction conditions and their relevance for brane cosmology. At the same time, this section is a good place to fix our notations and conventions and a few delicate signs.

3.2.1 Embedding of branes

Consider a D-dimensional Lorentzian manifold $\mathcal{M}^D$ with a non degenerate metric G. Locally, it is always possible to choose coordinates x^M, $M = 0, \cdots, D-1$. The corresponding metric components are denoted G_{MN}. On $\mathcal{M}^D$ imagine a number of intersecting or non intersecting hypersurfaces $\mathcal{N}_i$, $i = 1, \cdots, n$, which are submanifolds of $\mathcal{M}^D$. The number of co-dimensions of a submanifold is equal to $\dim(\mathcal{M}^D) - \dim(\mathcal{N}_i)$. In cosmological applications those submanifolds or hypersurfaces are world-sheets of branes, and therefore the number of co-dimensions is equal to the number of spatial extra-dimensions.

For practical purposes, it is useful to introduce locally a system of internal coordinates σ^μ on $\mathcal{N}_i$. An immersion[1] of $\mathcal{N}_i$ in $\mathcal{M}^D$ is given by the mapping

$$\begin{aligned} X^M : \mathcal{N}_i &\longrightarrow \mathcal{M}^D \\ \sigma^\mu &\longmapsto x^M \end{aligned} \tag{3.1}$$

with, for example, $x^0 = \sigma^0, \cdots, x^p = \sigma^p, x^{p+1} = 0, x^d = 0$, if $\mathcal{N}_i$ is a (p+1)-dimensional submanifold. Conversely, if x^M is given, it is always possible to choose locally internal coordinates σ^μ on $\mathcal{N}_i$ according to the prescription above, since there exist $p+1$ coordinate diffeomorphisms $\sigma \to \tilde{\sigma}(\sigma)$, which leave the Lagrangian invariant. Globally, this need not always be the case, for example if the branes are intersecting, or in a cluster of soap bubbles.

At each point q of $\mathcal{N}_i$ one can define the tangent space $\mathcal{T}_q\mathcal{N}_i$. A basis of $\mathcal{T}_q\mathcal{N}_i$ is given by the $p+1$ vectors $\frac{\partial}{\partial\sigma^0}, \cdots, \frac{\partial}{\partial\sigma^p}$. An arbitrary vector in $\mathcal{T}_q\mathcal{N}_i$ has then components

$$e^M_\mu = \frac{\partial X^M}{\partial \sigma^\mu}. \tag{3.2}$$

Knowing e^M_μ one can explicitly construct the internal[2] metric $g_{\mu\nu}$ on the brane $\mathcal{N}_i$ via the pull-back

$$g_{\mu\nu} = G_{MN} e^M_\mu e^N_\nu. \tag{3.3}$$

Suppose now that the action of the compound system of branes is the sum[3] $S = S_1 + \cdots + S_n$, where

$$S_i = \int \mathrm{d}^{p+1}\sigma \sqrt{-g}\, \mathcal{L}_i \tag{3.4}$$

is the action of a single brane. The Lagrangian density $\mathcal{L}_i$ depends only on fields living on the i-th brane, which, however, may be induced by external fields. The

[1] An immersion is a locally injective mapping.

[2] We shall use 'internal', 'intrinsic', and 'induced' metric as synonyms.

[3] This is for example the case in the Randall-Sundrum model with $n = 2$. Notice also that for this consideration we do not take into account possible bulk terms.

conserved Noether currents and the surface energy-momentum tensor of the i-th brane are

$$j^\mu = \frac{\partial \mathcal{L}_i}{\partial p_\mu}, \tag{3.5}$$

$$t^{\mu\nu} = -2\frac{\delta \mathcal{L}_i}{\delta g_{\mu\nu}} + \mathcal{L}_i g^{\mu\nu}, \tag{3.6}$$

for each canonical momentum p^μ. The symbol δ denotes a variational (Eulerian) derivative. There are other internal quantities, for example the Riemann tensor constructed from $g_{\mu\nu}$, if the action includes curvature terms. For specific computational purposes, it may be convenient to work with $g_{\mu\nu}, j^\mu$, and $t^{\mu\nu}$. However, if there are more or intersecting branes, the choice of internal coordinates is awkward. In this case, it is more advantageous to work with D-dimensional tensors of the background. One defines the D-dimensional energy momentum tensor by the push-forward

$$T^{MN} = t^{\mu\nu} e_\mu^M e_\nu^N. \tag{3.7}$$

This tensor is defined only at the location of the brane. In this way, all internal tensorial quantities can be lifted to become tensors of the full space-time $\mathcal{M}^D$. The calculations are then carried out with those background tensors. Further advantages of this approach are the absence of delta functions, which otherwise appear at the position of the branes, as well as conceptual simplicity. However in general, which view is more favorable to adopt, depends on the specific problem.

3.2.2 The first fundamental form

To construct T^{MN} according to Eq. (3.7), one must nevertheless specify internal coordinates σ^μ and an embedding to obtain the basis vectors e_μ^M. So in the end, using D-dimensional quantities would not be more economic or simple. There is an alternative way to find T^{MN}, or in fact any D-dimensional tensor, without falling back on the use of internal coordinates. Assume that $\mathcal{N}_i$ is a strictly time-like or space-like hypersurface. Then, at any point q of $\mathcal{N}_i$, it is possible to decompose the D-dimensional metric tensor into a part tangential and orthogonal to the surface element as

$$G_{MN} = q_{MN} + \perp_{MN}. \tag{3.8}$$

The tangential part q_{MN} is called first fundamental form. It is a D-dimensional tensor, but only defined on the hypersurface $\mathcal{N}_i$. The action of the mixed tensor $q^M{}_N$ is to project any D-dimensional tensor at a point $q \in \mathcal{N}_i$ onto the tangent space $\mathcal{T}_q\mathcal{N}_i$. Similarly, $\perp^M{}_N$ is an orthogonal projector. They satisfy

$$q^M{}_A q^A{}_N = q^M{}_N, \quad \perp^M{}_A \perp^A{}_N = \perp^M{}_N, \quad q^M{}_A \perp^A{}_N = 0. \tag{3.9}$$

For example, one can define a D-dimensional covariant derivative, that differentiates along the hypersurface, by

$$q^C{}_M \nabla_C \equiv {}^{||}\nabla_M. \tag{3.10}$$

Notice that ${}^{||}\nabla_M$ is not the covariant derivative associated with the internal metric $g_{\mu\nu}$, but only a tangential projection of the D-dimensional connection. We shall make extensive use of this in the next paragraph.

With the aid of q^{MN}, the D-dimensional tensor T^{MN} in Eq. (3.7) can be obtained without a detour on the induced metric simply by varying the Lagrangian with respect to the bulk metric

$$T^{MN} = -2\frac{\delta \mathcal{L}_i}{\delta G_{MN}} + \mathcal{L}_i q^{MN}. \tag{3.11}$$

In the second term, we have written q^{MN} instead of G_{MN}, since T^{MN} is equal to its own tangential part[4]. Calculations involving the energy-momentum tensor on the brane can now be carried out using this equivalent background quantity. Recall that the link with the original internal quantity is still given by Eq. (3.7).

An alternative definition of the first fundamental form is

$$q^{MN} = g^{\mu\nu} e^M_\mu e^N_\nu. \tag{3.12}$$

This is seen by writing Eq. (3.8) as

$$G^{MN} = q^{MN} + \perp^{MN} = g^{\mu\nu} e^M_\mu e^N_\nu + \perp^{MN}, \tag{3.13}$$

i.e. the full metric on $\mathcal{M}^D$ can be constructed by lifting the internal metric to a D-dimensional tensor, namely the metric along the submanifold $\mathcal{N}_i$, and adding the orthogonal complement. Here, also the difference between $q^M{}_N$ and e^M_μ becomes evident: e^M_μ promotes (pushes-forward) a tensor on $\mathcal{N}_i$ to a D-dimensional tensor and vice versa. In contrast, $q^M{}_N$ simply projects a D-dimensional tensors to its own tangential part. A useful relation for calculations (see for example paragraph 3.2.5) is

$$q^{MN} = G^{AB} q^M{}_A q^N{}_B, \tag{3.14}$$

which simply states that the projection of the D-dimensional metric yields the (also D-dimensional) metric along $\mathcal{N}_i$. In particular, q^{MN} can be used to raise and lower indices of D-dimensional tangential tensors.

3.2.3 The second fundamental form

At any point q of the p-dimensional submanifold $\mathcal{N}_i$, one can define the second fundamental form

$$K_{MN}{}^A = -q^B{}_N\, q^C{}_M \nabla_C q^A{}_B = -q^B{}_N\, {}^{||}\nabla_M q^A{}_B, \tag{3.15}$$

which is tangential to $\mathcal{N}_i$ in its first two indices and orthogonal in the last. The second fundamental form is the generalization of the extrinsic curvature tensor to an arbitrary number of co-dimensions. From the Weingarten equation, it follows

[4]The indices of the first fundamental form are pulled up and down with G_{MN}, as for any D-dimensional tensor.

that $K_{[MN]}{}^{A} = 0$, where the square bracket denotes antisymmetrization. By contracting over the tangential indices, one obtains the extrinsic curvature vector

$$K^A = K^M{}_M{}^A, \tag{3.16}$$

which is purely orthogonal onto $\mathcal{N}_i$, *i.e.* ${}^{||}K^B = q^B{}_A K^A$ is identically zero. The specification of K^A provides an equation of motion for the hypersurface. When there are no external forces, it takes the simple form $K^A = 0$. If the hypersurface is subject to an external force F^A, *e.g.* due to some gauge field in the bulk, the equation of motion is

$$T^{MN} K_{MN}{}^{A} = {}^{\perp}F^A. \tag{3.17}$$

The surface energy-momentum tensor T^{MN} is provided by Eq. (3.7) or (3.11), and ${}^{\perp}F_A = \perp^A{}_B F^B$ is the orthogonal projection of the external force. For instance, for a Nambu-Goto string (see Ref. [158]), $T_M{}^N = -\rho\,\delta_M{}^N$, and form $T_M{}^N K^M{}_N{}^A = {}^{\perp}F^A$ one finds

$$K^M{}_M{}^A = K^A = -\frac{1}{\rho}\,{}^{\perp}F^A. \tag{3.18}$$

In addition, there exists a formalism to treat perturbations on a hypersurface in terms of the first and second fundamental forms, see *e.g.* Ref. [16]. We shall make use of this in the case of one co-dimension in the article in Chap. 5.

3.2.4 First and second fundamental form for one co-dimension

So far, everything we have said is true for a hypersurface with an arbitrary number of co-dimensions. In brane cosmology, motivated by Hořawa-Witten compactification (see paragraph 2.5.3), one identifies our universe with a 3-brane embedded in a 5-dimensional bulk. We therefore make the identification, that this 3-brane corresponds to $\mathcal{N}_1$, and that the bulk is $\mathcal{M}^5$. Some models also require the presence of a second brane $\mathcal{N}_2$ which is aligned paralled to $\mathcal{N}_1$. In the following, however, we focus on a single brane and drop the subscript 1 to avoid cluttering the notation.

The unit normal $n^M \perp \mathcal{T}_q\mathcal{N}$ at the point $q \in \mathcal{N}$ is defined by the orthogonality and normalization conditions

$$\begin{aligned} &G_{MN} n^M e^N_\mu = 0, \quad \mu = 0, \cdots, p, \\ &G_{MN} n^M n^N = \pm 1. \end{aligned} \tag{3.19}$$

The plus sign applies for a space-like unit normal (time-like brane), and the minus sign for time-like unit normal (space-like brane). Causality requires that the trajectory of a brane in the bulk is time-like, and therefore we shall always assume the unit normal to be space-like, if not otherwise indicated[5]. The orthogonal metric tensor can now be simply expressed in terms of the unit normal

$$\perp_{MN} = n_M n_N, \tag{3.20}$$

[5]In the well-known $3+1$ splitting in general relativity, the unit normal is time-like. Therefore, care must be taken, when comparing our expressions with those in general relativity textbooks.

such that full space-time metric can be decomposed as

$$G_{MN} = q_{MN} + n_M n_N. \tag{3.21}$$

Also the second fundamental form can now be defined in terms of the unit normal

$$K_{MN} = -q^B{}_N q^C{}_M \nabla_C n_B = -q^B{}_N {}^{||}\nabla_M n_B. \tag{3.22}$$

Note that $\nabla_C n_B$ is in general not in the tangent space $\mathcal{T}_q\mathcal{N}$, therefore it needs to be projected onto $\mathcal{T}_q\mathcal{N}$ by the first fundamental form. The indices M and N are 'tangential', whereas B and C stand for arbitrary directions. K_{MN} is symmetric in M and N. In Gaussian normal coordinates[6] one has $n^C\nabla_C n_B = 0$, since the unit normal is tangential to a geodesic of $\mathcal{M}^5$, and Eq. (3.22) reduces to

$$K_{MN} = -\nabla_M n_N. \tag{3.23}$$

The second fundamental form is linked to the usual extrinsic curvature tensor $k_{\mu\nu}$, which measures 'extrinsically' the curvature of a hypersurface via the tilt of the unit normal as it is transported along the hypersurface. The relation is

$$K^{MN} = k^{\mu\nu} e^M_\mu e^N_\nu. \tag{3.24}$$

Like in Eq. (3.7), K^{MN} is defined only on the hypersurface $\mathcal{N}$. According to the specific problem, it is more convenient to work with $k_{\mu\nu}$ or K_{MN}. A definition of $k_{\mu\nu}$ consistent with (3.22) is

$$k_{\mu\nu} = -e^N_\mu e^M_\nu \nabla_M n_N. \tag{3.25}$$

Indeed,

$$\begin{aligned} K^{MN} &= k^{\mu\nu} e^M_\mu e^N_\nu = k^{\mu\nu} e^N_\mu e^M_\nu = -(e^\mu)^B (e^\nu)^C \nabla_C n_B \, e^N_\mu e^M_\nu \\ &= -\underbrace{(e^\mu)^B e^N_\mu}_{=q^{BN}} \underbrace{(e^\nu)^C e^M_\nu}_{=q^{CM}} \nabla_C n_B. \end{aligned} \tag{3.26}$$

Alternatively, $k_{\mu\nu}$ can be defined via the Lie derivative of the five-dimensional metric in the direction of the unit normal

$$k_{\mu\nu} = -\frac{1}{2} e^M_\mu e^N_\nu \mathcal{L}_n G_{MN}, \tag{3.27}$$

because by definition of the Lie derivative[7]

$$\begin{aligned} \mathcal{L}_n G_{MN} &= n^A G_{MN,A} + G_{AN} n^A{}_{,M} + G_{MA} n^A{}_{,N} \\ &= n^A G_{MN,A} + (n_{N,M} - G_{AN,M} n^A) + (n_{M,N} - G_{MA,N} n^A) \\ &= n_{N,M} + n_{M,N} - 2n^A \Gamma_{ANM} \\ &= \nabla_M n_N + \nabla_N n_M, \end{aligned} \tag{3.28}$$

[6] Gaussian normal coordinates are defined in the following way: from each point p in the neighborhood of $\mathcal{N}$, a geodesic in $\mathcal{M}^5$ is dropped onto $\mathcal{N}$, such that it intersects $\mathcal{N}$ orthogonally at a point q. p is then uniquely characterized by indicating the coordinates of q and a proper distance along the geodesic.

[7] We adopt the common, but untidy, notation $\mathcal{L}_n G_{MN}$, for what is acutally $(\mathcal{L}_n G)_{MN}$. Similarly, one writes $\nabla_M n_N$ instead of $(\nabla_M n)_N$.

where we have used the product rule and replaced metric derivatives by Christoffel symbols. Since the Lie derivative is symmetric, the definition (3.27) is indeed equivalent to $k_{\mu\nu} = -e_\mu^M e_\nu^N \nabla_M n_N$ given in Eq. (3.25).

A third equivalent definition is the following [150]: the extrinsic curvature is a symmetric bilinear form $K(.,.)$ acting on the tangent space $\mathcal{T}_q\mathcal{N}$, which satisfies

$$\nabla_X Y = \widehat{\nabla}_X Y + K(X,Y)n. \tag{3.29}$$

Here, $X, Y \in \mathcal{T}_q\mathcal{N}$ and $n \perp \mathcal{T}_q\mathcal{N}$. The symbol ∇ denotes the connection with respect to the metric G in $\mathcal{M}^5$, and $\widehat{\nabla}$ the connection for the internal metric g on $\mathcal{N}$, given by Eq. (3.3). By taking the scalar product with n on both sides,

$$(\nabla_X Y, n) = \underbrace{(\widehat{\nabla}_X Y, n)}_{=0} + K(X,Y)\underbrace{(n,n)}_{=1}, \tag{3.30}$$

one finds

$$K(X,Y) = (\nabla_X Y, n). \tag{3.31}$$

We now use the Ricci identity $XG(Y,Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z)$. The left hand side is zero, for ∇ is a metric connection, *i.e.* $\nabla G = 0$. Setting $Z = n$, one finds

$$K(X,Y) = -(Y, \nabla_X n), \tag{3.32}$$

which is equivalent to Eq. (3.23). Indeed, to write this equation in component form, we choose a basis $\{e_0, e_1, e_2, e_3, n\}$ of $\mathcal{T}_q\mathcal{M}^5$, and set $X = e_M, Y = e_N$ (both in the tangential space). Then

$$\begin{aligned} K_{MN} &= K(e_M, e_N) = -(e_N, \nabla_{e_M} n) = -(e_N, (\nabla_M n^L) e_L) \\ &= -(\nabla_M n^L)(e_N, e_L) = -\nabla_M n_N. \end{aligned} \tag{3.33}$$

3.2.5 The equations of Gauss, Codazzi and Mainardi

The Gauss equation (theorema egregium) provides a link between the intrinsic Riemann tensor on a submanifold with co-dimension one with the Riemann tensor of the embedding manifold. In brane cosmology, the Gauss equation is useful to find the Riemann, Ricci, and Einstein tensors on a brane from those in the bulk. For concreteness, we consider as before a time-like 4-dimensional submanifold $\mathcal{N}$, embedded in a 5-dimensional Lorentzian manifold $\mathcal{M}^5$. The Gauss theorem (theorema egregium) is [150]:

Theorem 4. *(Gauss)*

$$(R(X,Y)Z, W) = (\widehat{R}(X,Y)Z, W) + K(X,Z)K(Y,W) - K(Y,Z)K(X,W), \tag{3.34}$$

where $(R(X,Y)Z, W)$ denotes the Riemann tensor of $\mathcal{M}^5$, constructed from the metric G, and $\widehat{R}(X,Y)Z, W)$ is the Riemann tensor of $\mathcal{N}$, constructed from g given by the pull-back (3.3). All vector fields are tangential to the submanifold, *i.e.* $X, Y, Z, W \in \mathcal{T}_q\mathcal{N}$.

The components of the Gauss equation are obtained by setting $X = e_C, Y = e_D, Z = e_B, W = e_A$. Hence

$$\begin{aligned}(R(e_C, e_D)e_B, e_A) = (\widehat{R}(e_C, e_D)e_B, e_A) + K(e_C, e_B)K(e_D, e_A) \\ - K(e_D, e_B)K(e_C, e_A)\end{aligned} \tag{3.35}$$

or, equivalently,

$$R_{MNRS}q^M{}_A q^N{}_B q^R{}_C q^S{}_D = \widehat{R}_{ABCD} + K_{CB}K_{DA} - K_{DB}K_{CA}. \tag{3.36}$$

Notice that we have used bulk indices $M = 0, \cdots, 4$ according to the formalism introduced before. Since $\widehat{R}_{ABCD}$ is an internal brane quantity, it is clear that its indices actually run only from 0 to 3. The bulk Riemann tensor is evaluated at the position of the brane and tangentially projected[8] by $q^M{}_A q^N{}_B q^R{}_C q^S{}_D$. On the other hand, the extrinsic curvature is by definition located on the brane and tangentially projected onto it.

A complementary equation is that of Codazzi and Mainardi [150]:

Theorem 5.

$$(R(X, Y)Z, n) = (\widehat{\nabla}_X K)(Y, Z) - (\widehat{\nabla}_Y K)(X, Z), \tag{3.37}$$

where $X, Y, Z \in \mathcal{T}_q\mathcal{N}$ *and* $n \perp \mathcal{T}_q\mathcal{N}$.

In component language, with $n = n^A e_A$, it reads

$$n^A(R(e_C, e_D)e_B, e_A) = \widehat{\nabla}_C K_{DB} - \widehat{\nabla}_D K_{CB} \tag{3.38}$$

or, equivalently,

$$n^A R_{ANRS} q^N{}_B q^R{}_C q^S{}_D = \widehat{\nabla}_C K_{DB} - \widehat{\nabla}_D K_{CB}. \tag{3.39}$$

Here, only the last three indices of R_{ANRS} are projected onto the brane. Upon contraction with G^{BD}, a term $q^N{}_B q^S{}_D G^{BD} = q^{NS}$ appears on the left hand side, which is the 5-dimensional metric along $\mathcal{N}$, and therefore can be used to contract over the second and fourth index of the Riemann tensor. One easily finds

$$R_{AB}n^A q^B{}_C = \widehat{\nabla}_C K - \widehat{\nabla}_D K_C{}^D. \tag{3.40}$$

We have used the notation $K \equiv K_D{}^D$ for the trace.

The equations of Gauss, Codazzi, and Mainardi are useful to split the 5-dimensional Einstein tensor,

$$\mathcal{G}_{AB} = R_{AB} - \frac{1}{2}G_{AB}R, \tag{3.41}$$

[8]Alternatively, we could have written ${}^{||}R_{ABCD}$ for the left hand side.

into 'mixed', 'orthogonal', and 'parallel' components with respect to the timelike hypersurface $\mathcal{N}$ (see Ref.[19]). The mixed component corresponds directly to Eq. (3.40), as the second term in Eq. (3.41) does not contribute upon contraction with $q^B{}_C$. Thus

$$\mathcal{G}_{AB} n^A q^B{}_C = \widehat{\nabla}_C K - \widehat{\nabla}_D K_C{}^D. \tag{3.42}$$

Similarly, the orthogonal component is

$$\mathcal{G}_{AB} n^A n^B = -\frac{1}{2}\widehat{R} + \frac{1}{2}\left(K^2 - K_{AB}K^{AB}\right), \tag{3.43}$$

where $\widehat{R}$ denotes the Riemann scalar constructed from the internal metric $g_{\mu\nu}$. Finally, the parallel component is

$$\begin{aligned}\mathcal{G}_{AB} q^A{}_C q^B{}_D = \widehat{\mathcal{G}}_{CD} - K K_{CD} + n^E \nabla_E K_{CD} + \widehat{\nabla}_C a_D + 2n_{(C} K_{D)E} a^E - a_C a_D \\ + \Big(\frac{1}{2}K^2 + \frac{1}{2}K_{AB}K^{AB} - n^A \nabla_A K - \nabla_B a^B\Big) q_{CD},\end{aligned} \tag{3.44}$$

where $\widehat{\mathcal{G}}_{CD}$ is the Einstein tensor constructed from the internal metric, $a^B = n^C \nabla_C n^B$ is the 'acceleration field', and the round brackets denote symmetrization with weight one half. Notice also the difference between the two covariant derivatives that appear in the above formula: ∇ associated with the metric G_{MN} and $\widehat{\nabla}$ associated with $g_{\mu\nu}$.

3.2.6 Junction conditions

So far, we have considered $\mathcal{N}$ to be an empty hypersurface. In a cosmological context, however, it corresponds to a brane with a certain energy density due to matter and radiation. The energy content has an effect on the bulk spacetime, and therefore the geometry on both sides must be matched in a proper way. The problem is somewhat subtle, because the smoothness of the bulk metric is influenced not only by the presence of the brane, but also by the particular choice of coordinates to describe the bulk[9]. An example is the treatment of thin shells in ordinary 4-dimensional general relativity. In Ref. [82] a formalism was developed, making no reference to coordinate systems, in order to study boundary surfaces or shock waves, where the density jumps, as well as surface layers, where it becomes infinite. Earlier work on the subject has been done by Refs. [99], [141], and [45]. The same formalism is equally suited for brane cosmology.

We want to know about the local geometry due to some matter distribution. Therefore we start with the 5-dimensional Einstein equations written in the form

$$R_{AB} = \kappa_5^2 \left(T_{AB} - \frac{1}{3}\, G_{AB} T\right), \tag{3.45}$$

[9] In Newtonian theory, the solution of a matching problem is straightforward: there is no problem in setting up a well-defined coordinate system, and one simply has to impose continuity and jump conditions for the Newtonian potential and its first derivative across the hypersurface.

where $T \equiv T^A{}_A$ denotes the trace of the 5-dimensional energy-momentum tensor. The unusual factor 1/3 arises because $G_{AB}G^{AB} = 5$ rather than 4. Assume that the brane has a finite thickness ε, and let $\mathcal{N}^<$ and $\mathcal{N}^>$ denote the two boundaries of $\mathcal{N}$. A Gaussian normal coordinate y is defined such that $y = 0$ on $\mathcal{N}^<$ and $y = \varepsilon$ on $\mathcal{N}^>$. In Gaussian normal coordinates, the 5-dimensional Ricci tensor, projected onto the brane, can be written as [82]

$$R_{CD}q^C{}_A q^D{}_B = \frac{\partial K_{AB}}{\partial y} + \widehat{R}_{AB} - KK_{AB} + 2K_A{}^C K_{CB}. \tag{3.46}$$

This form of the Ricci tensor is inserted into the Einstein equations (3.45), and both sides are integrated from $\mathcal{N}^<$ to $\mathcal{N}^>$,

$$\begin{aligned}\int_0^\varepsilon \mathrm{d}y \frac{\partial K_{AB}}{\partial y} + \int_0^\varepsilon \mathrm{d}y \left(\widehat{R}_{AB} - KK_{AB} + 2K_A{}^C K_{CB}\right) \\ = \kappa_5^2 \int_0^\varepsilon \mathrm{d}y \Big(T_{CD}q^C{}_A q^D{}_B - \frac{1}{3}\underbrace{G_{CD}q^C{}_A q^D{}_B}_{=q_{AB}} T\Big).\end{aligned} \tag{3.47}$$

Notice that, on the left hand side, all tensors are by definition tangential, whereas on the right hand side the bulk energy-momentum tensor and the bulk metric have to be projected onto each hypersurface in the integration domain. It is reasonable to expect that the second term on the left hand side is bounded between $\mathcal{N}^<$ and $\mathcal{N}^>$, and therefore the integral vanishes in the limit $\varepsilon \to 0$.

We define the surface energy-momentum tensor of $\mathcal{N}$ by

$$S_{AB} \equiv \lim_{\varepsilon\to 0} \int_0^\varepsilon \mathrm{d}y\, T_{CD}q^C{}_A q^D{}_B, \tag{3.48}$$

where $S = S_{AB}q^{AB}$ denotes the trace. As a first junction condition, we assume that the first fundamental form is continuous across $\mathcal{N}$, *i.e.*

$$q^>_{AB} = q^<_{AB}. \tag{3.49}$$

In the limit of an infinitely thin brane, Eq. (3.47) then becomes

$$K^>_{AB} - K^<_{AB} = \kappa_5^2 \left(S_{AB} - \frac{1}{3}\, q_{AB} S\right). \tag{3.50}$$

In the literature, this is known as Israel's junction condition[10]. The relations (3.49) and (3.50) guarantee the consistent matching of the bulk geometry across the brane. In brane cosmology, motivated by the Hořava-Witten compactificaton, one often works with a Z_2-symmetric bulk, *i.e.* with a symmetry under $y \to -y$. Then, $K^<_{AB} = -K^>_{AB} \equiv -K_{AB}$, and Eq. (3.50) becomes[11]

$$K_{AB} = \frac{\kappa_5^2}{2}\left(S_{AB} - \frac{1}{3}\, q_{AB} S\right), \tag{3.51}$$

[10] Often they are written in terms of internal brane quantities, namely $k^>_{\mu\nu} - k^<_{\mu\nu} = \kappa_5^2\left(S_{\mu\nu} - \frac{1}{3}g_{\mu\nu}S\right)$, where $S^\mu{}_\nu = \mathrm{diag}(-\rho, P, P, P)$, for example

[11] The choice of the 'right' side of the brane as representative for the whole bulk can be subtle. See the discussion in the article 'On CMB anisotropies from vector perturbation in the bulk'.

i.e. the extrinsic curvature of the brane is completely determined by its matter content.

The power of this approach lies in the fact that it makes no reference to a particular coordinate system, and that no knowledge is needed about the global structure of the manifold $\mathcal{M}^5$. The basic formalism is valid for time-like (as assumed here) and space-like hypersurfaces, but not for surfaces that are null, since then the concept of extrinsic curvature breaks down.

In this section, we have introduced various tensors to describe the geometry of hypersurfaces, both from the full space-time point of view (q_{AB}, K_{AB}), and from an intrinsic point of view ($g_{\mu\nu}, k_{\mu\nu}$). We have shown how they are related to each other, and how they can be used for practical calculations. Now, our aim is to motivate the existence of such hypersurfaces by introducing p-branes, which are physical objects arising in supergravity and string theory.

3.3 Branes from supergravity

3.3.1 The black p-brane geometry

Originally, p-branes were found as classical solutions to supergravity. Since 11-dimensional supergravity, compactified on a circle, is equivalent to the low energy limit of 10-dimensional type IIB string theory, we can start from the equations of motion (2.7). We are looking for solutions that correspond to a p-dimensional source of charge Q_p under the 4-form potential C_4. The metric is required to have p-dimensional Euclidean symmetry and, for simplicity, spherical symmetry in the remaining $10-p$ dimensions. The spherical part, including time, has Lorentzian signature. In the string frame, the solution for the metric is [2]

$$\mathrm{d}s_{10}^2 = -\frac{f_+(\rho)}{\sqrt{f_-(\rho)}}\mathrm{d}t^2 + \sqrt{f_-(\rho)}\sum_{i=1}^{p}\mathrm{d}x^i\mathrm{d}x^i + \frac{f_-(\rho)^{\varrho}}{f_+(\rho)}\mathrm{d}\rho^2 + \rho^2 f_-(\rho)^{\varsigma}\mathrm{d}\Omega_{8-p}^2, \quad (3.52)$$

where ρ is the radial coordinate of the spherical part, x^i are the spatial coordinates along the source, and $\mathrm{d}\Omega_{8-p}^2$ is the volume element of an $8-p$ sphere. Furthermore, we have defined

$$f_\pm(\rho) = 1 - \left(\frac{r_\pm}{\rho}\right)^{7-p}, \quad \varrho = -\frac{1}{2} - \frac{5-p}{7-p}, \quad \varsigma = \frac{1}{2} - \frac{5-p}{7-p}. \quad (3.53)$$

The solution for the dilaton field is

$$\mathrm{e}^{-2\Phi} = g_s^{-2} f_-(\rho)^{-\frac{p-3}{2}}. \quad (3.54)$$

Since $f_-(\rho) = 1$ in the limit $\rho \to \infty$, g_s represents the asymptotic value of the string coupling constant. For $p = 3$, the dilaton is constant.

Eq. (3.52) is called black p-brane solution, in analogy with the charged black-hole or Reissner-Nordström geometry. In particular, it has an inner and outer horizon, r_- and r_+. When $r_+ > r_-$, the singularity is always shielded by the horizon, and the p-brane can be regarded as a black-hole. In analogy with the Reissner-Nordström case, this corresponds to the mass (density) of the source being larger than its charge (density), $M_p > Q_p$.

In a string theoretical context, the source is formed by a stack of N Dp-branes on top of each other. The charge of a Dp-brane is equal to its tension τ_p, given in Eq. (3.73), such that $Q_p = N\tau_p$. When $r_+ < r_-$, there is a time-like naked singularity, which corresponds to the unphysical regime $M_p < Q_p$ in the Reissner-Nordström case. Finally, if $r_- = r_+$, the lower bound on the mass is saturated, $M_p = Q_p$. Such a solution is referred to as extremal p-brane, and it corresponds to a ground state. For $p \neq 3$, the horizon and the singularity coincide, and the supergravity description for an extremal p-brane breaks down close to the horizion. Then, one has has to resort to the full string theory description. For $p = 3$, however, the surface $\rho = r_- = r_+$ corresponds to a regular horizon, and it is this case that is particularly interesting for cosmology, if our universe is identified with a 3-brane moving in the background of a black 3-brane solution.

To discuss extremal and non extremal solutions with $r_+ \geq r_-$, we introduce a parameter r_H measuring the departure from extremality,

$$r_H^{7-p} = r_+^{7-p} - r_-^{7-p}, \tag{3.55}$$

and a new radial coordinate r,

$$r^{7-p} = \rho^{7-p} - r_-^{7-p}, \tag{3.56}$$

which is suited to describe the region outside the inner horizon. Note that the p-brane source is located at $\rho = 0$, whereas $r = 0$ denotes the location of the inner horizon. We also define

$$H_p(r) = 1 + \left(\frac{r_-}{r}\right)^{7-p}, \quad F_p(r) = 1 - \left(\frac{r_H}{r}\right)^{7-p}, \tag{3.57}$$

where H_p is a harmonic function. Then

$$f_- = \frac{1}{H_p}, \quad f_+ = \frac{F_p}{H_p}. \tag{3.58}$$

Inserting Eqs. (3.58) into the solution (3.52), and using that $r^{7-p} + r_-^{7-p} = H_p\, r^{7-p}$ and $\mathrm{d}\rho = \frac{r^{6-p}}{\rho^{6-p}}\mathrm{d}r$, the black p-brane geometry becomes

$$\mathrm{d}s_{10}^2 = H_p^{-1/2}\left(-F_p(r)\mathrm{d}t^2 + \sum_{i=1}^{p}\mathrm{d}x^i\mathrm{d}x^i\right) + H_p^{1/2}\left(\frac{\mathrm{d}r^2}{F_p(r)} + r^2\mathrm{d}\Omega_{8-p}^2\right). \tag{3.59}$$

The metric (3.59) is completely equivalent to the original metric (3.52) in the region outside the inner horizon. In the extremal limit, $r_H = 0$ and $F_p = 1$, and the Euclidean symmetry is enhanced to Poincaré symmetry. Eq. (3.59) describes the geometry induced by a p-dimensional source of mass M_p and charge Q_p. These two parameters can be expressed in terms of r_- and r_+, and are therefore contained in the function $H_p(r)$ and $F_p(r)$. For our purpose, they are not relevant, as we will be merely interested in a particular limit of the metric (3.59), namely an anti-de Sitter-Schwarzschild space-time (AdS$_5$-S) which we are going to derive in the next paragraph. Nevertheless, we find it instructive to see how the parameters of AdS$_5$-S are linked to the fundamental supergravity solution.

Notice that p-branes are solitons of supergravity. When making the analogy with topological defects, a 0-brane corresponds to a monopole, a 1-branes to a cosmic string, and a 2-brane to a domain wall. The peculiar case $p = -1$ is an instanton. On the other hand, there is a close correspondence to string theory D-branes. We will comment on this and on the validity of the supergravity solution in Sec. 3.4.

3.3.2 Anti-de Sitter space-time

The curvature of the geometry (3.59) is set by the horizon r_-, which in turn is determined by the mass and the charge of the p-brane source. It is interesting to consider the black p-brane solution in the limit $r \ll r_-$. Because $r = 0$ corresponds to the horizon itself, this limit means being near the horizon. Therefore, the limit $r \ll r_-$ is commonly called the near horizon limit. Let us define the variable $L \equiv r_-$ for future convenience, and set $p = 3$. This choice will be justified below. Essentially, the near horizon limit then corresponds to the approximation

$$H_3(r) = 1 + \left(\frac{L}{r}\right)^4 \simeq \left(\frac{L}{r}\right)^4 . \tag{3.60}$$

The metric (3.59) then takes the form

$$\begin{aligned} \mathrm{d}s_{10}^2 &= -\frac{r^2}{L^2}\left(1-\frac{r_H^4}{r^4}\right)\mathrm{d}t^2 + \frac{r^2}{L^2}\sum_{i=1}^{3}\mathrm{d}x^i\mathrm{d}x^i + \frac{\mathrm{d}r^2}{\frac{r^2}{L^2}\left(1-\frac{r_H^4}{r^4}\right)} + L^2\mathrm{d}\Omega_5^2 \\ &\equiv -f(r)\mathrm{d}t^2 + g(r)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^i + \frac{\mathrm{d}r^2}{f(r)} + L^2\mathrm{d}\Omega_5^2. \end{aligned} \tag{3.61}$$

Thus, the 10-dimensional space-time splits into $\mathcal{M}^5 \times S^5$, where $\mathcal{M}^5$ is a manifold with anti-de Sitter-Schwarzschild (AdS$_5$-S) geometry and S^5 is a 5-sphere. $\mathcal{M}^5$ is a non compact space-time, labelled by coordinates t, x^1, x^2, x^3, r, and contains a 3-dimensional Euclidean subspace. The volume element of the 5-sphere is given by

$$\mathrm{d}\Omega_5^2 = \mathrm{d}\theta_1^2 + \sin^2(\theta_1)\left\{\mathrm{d}\theta_2^2 + \sin^2(\theta_2)\left[\mathrm{d}\theta_3^2 + \sin^2(\theta_3)\left(\mathrm{d}\theta_4^2 + \sin^2(\theta_4)\mathrm{d}\theta_5^2\right)\right]\right\}. \tag{3.62}$$

Moreover, we have defined metric functions $f(r)$ and $g(r)$ for later convenience (see. Chap. 5). Here, the parameter r_H, which classified before whether the solution is extremal ($r_H = 0$) or non extremal ($r_H > 0$) plays the role of a horizon. In fact, without knowing the supergravity origin, one could interpret the geometry (3.61) as arising from a black-hole at $r = 0$.

Let us now consider the extremal limit $r_H = 0$ of the metric (3.61). Then, $\mathcal{M}^5$ becomes pure 5-dimensional anti-de Sitter space-time (AdS_5), and together with the spherical part, this is called the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry. The metric of the AdS_5 part is

$$\mathrm{d}s_5^2 = \frac{r^2}{L^2}\left(-\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) + \frac{L^2}{r^2}\mathrm{d}r^2. \tag{3.63}$$

In summary, we have shown that AdS_5 arises from the black 3-brane supergravity solution by taking both the near horizon limit and the extremal limit. The inner horizon r_- has become the curvature radius L of AdS_5, since by definition $L = r_-$. Of course, anti-de Sitter space-time exists also on its own right as a maximally symmetric solution of Einstein's equations with a negative cosmological constant (see Sec. 2.4).

In brane world cosmology, AdS_5 is particularly important due to the generalized Birkhoff theorem [26]. Most brane world models consider an AdS_5 background, and our universe is identified with a 3-brane moving in it. The observed homogeneity and isotropy can be accommodated by choosing an embedding in which the brane is parallel to the Minkowskian part of the metric (3.63). This also explains why we have chosen $p = 3$ at the beginning.

We end this section by giving some alternative forms of the AdS_5 metric (3.63) which are useful in various calculations. First, by defining a new radial coordinate $z \equiv L^2/r$, the metric (3.63) can be cast into the form

$$\mathrm{d}s_5^2 = \frac{L^2}{z^2}\left(-\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}z^2\right). \tag{3.64}$$

Alternatively, one may work with a radial coordinate $\varrho \equiv -L\ln(r/L)$, such that $r^2/L^2 = \mathrm{e}^{-2\varrho/L}$. In these coordinates the metric (3.63) reads

$$\mathrm{d}s_5^2 = \mathrm{e}^{-2\varrho/L}\left(-\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) + \mathrm{d}\varrho^2, \tag{3.65}$$

which is the form used in the so-called Randall-Sundrum model: a Minkowski metric on a 3-brane is multiplied by an exponentially decreasing 'warp' factor.

Finally, AdS_5 space-time can be understood as a hyperboloid embedded in a 6-dimensional flat space with coordinates $(x_0, x_1, x_2, x_3, x_4, x_5)$ and a metric

$$\mathrm{d}s_6^2 = -\mathrm{d}x_0^2 - \mathrm{d}x_5^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2 \tag{3.66}$$

subject to the constraint

$$-x_0^2 - x_5^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = -\mathrm{const}^2. \tag{3.67}$$

3.4 Branes from string theory

In string theory there exist objects called Dp-branes, which are hypersurfaces representing the geometrical locus where open strings end. The letter 'D' refers to Dirichlet boundary condition, and 'p' stands for the number of spatial dimensions as we are familiar with from the previous section. Their existence was first claimed in Ref. [126]. In this section we follow however the more pedagogic treatment of [124], [125], and [83].

3.4.1 T-duality for open strings

To see how D-branes come about, recall the mode expansion for the open string, Eq. (2.10). An open string supports oscillatory modes travelling from its left to its right end, and vice versa. To make this evident, one rewrites the solution (2.10) as

$$X^M(\tau,\sigma) = x^M + \frac{p^M}{p^+}\tau + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n\neq 0}\frac{\alpha_n^M}{n}\left(\mathrm{e}^{-\pi n(ic\tau+i\sigma)/s} + \mathrm{e}^{-\pi n(ic\tau-i\sigma)/s}\right). \tag{3.68}$$

The first term in the oscillatory modes is due to left-, the second due to right-movers. By the Neumann boundary condition they are not independent, therefore there is just one amplitude α_n^M for both. Note that we have replaced the index i in Eq. (2.10) by $M = 0,\cdots,D-1$ because, after all, the expansion is valid for all directions. We proceed now similarly as in paragraph 2.3.3: by an analytic continuation to a Minkowskian world-sheet, one defines a real coordinate $\sigma_0 = ic\tau$ and a complex variable $z = \mathrm{e}^{\sigma_0+i\sigma}$. The time τ can then be expressed as $\tau = \ln(|z|^2)/(2ic)$. This time we chose the length of the (open) string to be $s = \pi$. Hence the velocity defined below Eq. (2.9) becomes $c = 1/(2\alpha' p^+)$. In terms of the complex variables z and its conjugate $\bar{z}$, Eq. (3.68) reads

$$\begin{aligned} X^M(z,\bar{z}) &= X^M(z) + X^M(\bar{z}) \\ &= \left(\frac{x^M}{2} + \frac{x'^M}{2} - i\alpha' p^M\ln(z) + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n\neq 0}\frac{\alpha_n^M}{n}z^{-n}\right) \\ &\quad + \left(\frac{x^M}{2} - \frac{x'^M}{2} - i\alpha' p^M\ln(\bar{z}) + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n\neq 0}\frac{\alpha_n^M}{n}\bar{z}^{-n}\right). \end{aligned} \tag{3.69}$$

This splitting is analogous to Eq. (2.21) and attributes a holomorphic function to the left-movers and an anti-holomorphic function to the right-movers. We have added a coordinate x'^M whose meaning will soon become obvious. Assume that one direction, x^d say, is a circle with radius R. One can reparameterize the compact direction in terms of a so-called T-dual coordinate x'^d defined by

$$\begin{aligned} X'^d(z,\bar{z}) &= X^d(z) - X^d(\bar{z}) \\ &= x'^d + 2\alpha' p^d \sigma + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{\alpha_n^d}{n}\left(\mathrm{e}^{-n(ic\tau+i\sigma)} - \mathrm{e}^{-n(ic\tau-i\sigma)}\right) \\ &= x'^d + 2\alpha'\left(\frac{n}{R}\right)\sigma + (2\alpha')^{1/2}\sum_{n\neq 0}\frac{\alpha_n^d}{n}\mathrm{e}^{-inc\tau}\sin(n\sigma). \end{aligned} \tag{3.70}$$

Here, x'^d is the center of mass in terms of the T-dual coordinate.

Along a compact direction the momentum is quantized according to $p^d = n/R,\ n \in \mathbb{Z}$. However, there is no time dependence of the zero mode and, in addition, since the oscillator terms vanish at $\sigma = 0, \pi$, the endpoints of the open string do not move in the x'^d direction. From Eq. (3.70) one sees that the T-dual coordinate respects Dirichlet boundary conditions $\partial_\tau X'^d = 0$ at $\sigma = 0, \pi$, whereas in the other non compact directions, one still has Neumann boundary conditions $\partial_\sigma X^M = 0, M \neq d$. If $D = 10$ say, the open string ends are still free to move along eight spatial dimensions thereby defining a 8-dimensional hypersurface, a D8-brane. Note that translational invariance in x'^d direction is broken.

The distance between the open string endpoints in the direction x'^d is

$$X'^d(\sigma=\pi) - X'^d(\sigma=0) = 2\alpha'\left(\frac{n}{R}\right)\pi \equiv 2\pi R'\omega, \tag{3.71}$$

where $R' = \alpha'/R$ is the radius of the circle in the T-dual coordinate, and $\omega = n$ now denotes a winding number. The interpretation of Eq. (3.71) is that for $\omega = 0$ an open string begins and ends on the same D-brane without winding around the compact direction, while for $\omega \neq 0$, the string winds around ω times before coming back (see Fig. 3.1).

The existence of winding modes is an intrinsically stringy phenomenon and will be of great importance in string gas and brane gas cosmology (part III of this thesis). We end this paragraph by stressing the crucial result that, under a T-duality transformation, lengths are inverted and momentum modes becomes winding mode and vice versa

$$R \longleftrightarrow \frac{\alpha'}{R}, \quad n \longleftrightarrow \omega. \tag{3.72}$$

3.4.2 Some properties of D-branes

So far we have T-dualized only in one direction. One may also compactify other directions and repeat the procedure described above for the corresponding coordinates. Thereby, one successively generates a D7-brane, a D6-brane and so on. A D1-brane is also called D-string to distinguish it from a fundamental string. (As a difference, for instance, D-strings are much heavier than fundamental strings at small string coupling.)

Figure 3.1: The horizontal axis corresponds to the T-dual coordinate x'^d. An open string with winding number zero starts and ends on the same Dp-brane, whereas an open string with single winding can be represented by identifying the hyperplanes on the left and on the right. Note that those are not Dp-branes; they just symbolize the periodic identification.

Let us take a closer look at the action of T-duality. Along the brane, the open string ends are subject to the Neumann boundary conditions. Upon T-dualizing in a tangential direction, an additional Dirichlet boundary condition arises, and a Dp-brane is turned into a D(p-1)-brane. Similarly, a T-duality in an orthogonal direction turns a Dp-brane into a D(p+1)-brane. T-dualizing twice in the same direction yields back the original brane. The effect of T-duality on various winding states is discussed in detail in Sec. 11.4.

For the theory to be consistent, one finds that the dimensionalities of D-branes must not be arbitrary. In type IIA string theory the allowed branes are those with even p, hence D0, D2, D4, D6, D8-branes, while in type IIB the brane must have odd p, hence there are D1, D3, D5, D7, D8-branes.

As a next issue, we remark that the open string ends can be endowed with degrees of freedom whose associated wave functions are called Chan-Paton factors. Imagine a number N of D-branes on top of each other, while the ends of a string connect some pair in the stack. The end-states are labelled $|i\rangle$ and $|j\rangle$, $i = 1, \cdots, N$, indicating the branes on which the string begins and ends. Those states play the role of gauge charges. In a regime where $g_s N \ll 1$, perturbative field theory is valid, and the dynamics on the stack of branes are described by a $U(N)$ gauge theory [2]. This is interesting if the stack is identified with our universe: gauge particles are confined to the brane world, and therefore forces mediated by them are not sensitive to the existence of extra-dimensions. In contrast, the graviton is an excitation of a closed string, which is not bound to any brane, and hence gravity feels all space dimensions. This observation could reconcile the postulate of 'large' extra-dimensions (see Sec. 2.7) with the fact that there is no evidence for extra-dimensions from collider experiments probing electromagnetic, weak and strong forces down to $1/(200\,\mathrm{GeV})$.

A Dp-brane is not just a hyperplane, but also a physical object with localized energy and certain conserved quantum numbers: it is a soliton in string theory. In particular, any Dp-brane has a (positive) tension

$$\tau_p = \frac{1}{g_s \alpha'^{(p+1)/2}}, \tag{3.73}$$

which can be obtained from the amplitude of closed string exchange between two parallel Dp-branes. Since closed strings give rise to the graviton, this interaction is attractive. The tension (3.73) is proportional to the inverse of the string coupling constant whatever the value of p. (This is a non perturbative effect.)

At the same time, a Dp-brane carries a charge ρ_p, which couples to a Ramond-Ramond (p+1)-form field as discussed in Sec. 2.2. Conversely, each Dp-brane is the source of a gauge field in the bulk. Since Dp-branes have the same sign of charge, the interaction is repulsive. The sum of the two forces is zero if the Dp-branes are parallel to each other, and if they are BPS states, *i.e.* their tension (mass) is equal to their charge, $\tau_p = \rho_p$. In this case, there is no net force between the branes, and the system is stable. We investigate BPS and non BPS configurations in paragraph 5.2.2 by means of an effective potential, and discuss how instabilities of the latter translate into cosmological perturbations.

It is believed that Dp-branes and p-branes are different descriptions of the same object. The p-branes are classical solutions of supergravity, whereas the Dp-branes arise from a theory taking quantum effects into account. They are thus the full string theoretical version of p-branes. This interpretation fits well in the picture, that supergravity is the low energy limit of string theory [2]. The supergravity description is valid, if the curvature of the p-brane geometry, set by the horizon r_-, is small compared to the curvature defined by the string length $\ell_s = \alpha'^{1/2}$, *i.e.* $r_- \gg \ell_s$, such that stringy corrections are negligible.

We also remark that the classical parameters r_-, r_+ can be expressed in terms of string theory quantities such as g_s and N, and that the comparison of p-branes and Dp-branes has led to the discovery of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence. For more fundamental issues concerning Dp-branes we refer the reader to [125].

Finally, we remark that there is an action governing the dynamics of Dp-branes: this so-called the Dirac-Born-Infeld (DBI) action is the string theory generalization of the Nambu-Goto action for topological defects. We introduce it in the next chapter.

After these preliminaries on extra-dimensions and branes, we now turn to main subject of this thesis which is brane cosmology. Henceforth, we shall use p-branes and Dp-branes in a cosmological context.

Part II

BRANE COSMOLOGY

Chapter 4

Cosmology on a probe brane

4.1 Introduction

In chapters 2 and 3 we introduced the idea of extra-dimensions and extended objects, generalizing the physics of point particles in 4-dimensional space-time to that of p-branes in D-dimensional space-time. We also mentioned the idea that in this context our universe may be a 3-brane embedded in a higher-dimensional space-time. In this chapter, we are going to present a simple model which makes a natural link between string theory and cosmology.

Let us consider a 10-dimensional space-time, whose geometry is given by a low energy solution of string theory. We take this to be a 5-dimensional anti-de Sitter-Schwarzschild geometry multiplied by a 5-sphere (AdS_5-S$\times$S^5), as derived in Sec. 3.3. This geometry describes the gravitational field generated by a stack of 3-branes in the near horizon limit. Suppose now that our universe is another 3-brane moving along a geodesic in this curved background. Consequently, some dynamics are generated on the brane, even though the background itself is static. In fact, the motion through the higher-dimensional space-time induces a cosmological evolution on the brane, mimicking the effect of matter that usually drives the expansion. In this scenario, there is no real matter on the brane, and in allusion to the illusion, this scenario is called the 'mirage' cosmology.

We thus consider a 10-dimensional non-compact space-time $(\mathcal{M}^{10}, G)$ with coordinates $x^M = (t, x^1, x^2, x^2, r, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ and a metric

$$\mathrm{d}s_{10}^2 = G_{MN}\mathrm{d}x^M\mathrm{d}x^N = -f(r)\mathrm{d}t^2 + g(r)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + h(r)\mathrm{d}r^2 + s(r)\mathrm{d}\Omega_5^2. \quad (4.1)$$

For $h = 1/f, s = L^2$ and where f and g are the functions given in Eq. (3.61), this is the AdS_5-S$\times$S^5 metric, or in the extremal limit $\mathrm{AdS}_5\times$S^5. For the time being, we just assume that the metric coefficients f, g, h, s are some arbitrary functions of the radial coordinate r, in order to accommodate also other string theory backgrounds. With this ansatz, there is a Euclidean symmetry along the 3-brane and a spherical symmetry in the other spatial directions. We shall write out the volume element of the 5-sphere as $\mathrm{d}\Omega_5^2 = s_{IJ}\mathrm{d}\theta^I\mathrm{d}\theta^J$ where $I, J = 6, \cdots, 10$ (see also Eq. (3.62)). The generalization for arbitrary p is straightforward: in this case, there is a p-dimensional Euclidean subspace, and the spherically symmetric part is represented by a $8 - p$ sphere.

We investigate the geodesic motion of a 3-brane (to be identified with our universe) in the background (4.1). This brane is assumed to be 'light', such that the back-reaction onto the ambient geometry can be neglected. In analogy with a test particle, we shall often refer to it as a test or probe brane. Throughout this chapter we follow closely Ref. [90], in which this idea of mirage cosmology was first developed.

4.2 Probe brane dynamics

Our aim is to derive an equation of motion for the brane which, in the end, will translate into a Friedmann-like equation on the brane. A simple way of doing this, is to start from the Dirac-Born-Infeld (DBI) action which governs the dynamics of a Dp-brane, see Ref. [90]. The action we consider is (with $p = 3$)

$$S_{D3} = -T_3 \int \mathrm{d}^4\sigma\, \mathrm{e}^{-\Phi}\sqrt{-g} - T_3 \int \mathrm{d}^4\sigma\, \widehat{C}_4. \tag{4.2}$$

The first term is a special case of the full DBI action (see Eq. (5.1)), where we have neglected the antisymmetric tensor field $B_{\mu\nu}$ and the gauge field $F_{\mu\nu}$. Here, T_3 is the brane tension, σ^μ are the coordinates on the brane worldsheet, Φ is the 10-dimensional dilaton, and g is the determinant of the induced metric. The second term is a Wess-Zumino term describing the coupling of the 3-brane to the Ramond-Ramond 4-form field in the bulk. The quantity $\mathrm{d}^4\sigma\widehat{C}_4$ under the integral represents the pull-back of this 4-form onto the brane.

To calculate the induced metric on the 3-brane, we need to specify an embedding $x^M = X^M(\sigma^\mu)$. The requirements of isotropy and homogeneity on the brane are met by choosing

$$X^0 = t,\ X^i = x^i,\ X^4 = R(t),\ \theta^I = \Theta^I(t). \tag{4.3}$$

Then, the induced metric is

$$\mathrm{d}s_4^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \left(-f(R) + h(R)\dot{R}^2 + s(R)s_{IJ}\dot{\Theta}^I\dot{\Theta}^J\right)\mathrm{d}t^2 + g(R)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{4.4}$$

the dot denoting derivative with respect to t. From the DBI action, one reads off the Lagrangian density

$$\begin{aligned}\mathcal{L} &= -\mathrm{e}^{-\Phi}\sqrt{-g} - \widehat{C}_4 \\ &\equiv -\sqrt{A(R) - B(R)\dot{R}^2 - D(R)\dot{\Theta}^2} - C(R),\end{aligned} \tag{4.5}$$

where we have cast the determinant of the metric g, and the exponential prefactor $\mathrm{e}^{-\Phi}$ into the functions $A(R), B(R), D(R)$, to be evaluated at the location of the brane $r = R(t)$, where

$$\begin{aligned}A(R) &\equiv \mathrm{e}^{-2\Phi}f(R)g(R)^3, \\ B(R) &\equiv \mathrm{e}^{-2\Phi}g(R)^3h(R), \\ D(R) &\equiv \mathrm{e}^{-2\Phi}g(R)^3s(R).\end{aligned} \tag{4.6}$$

Furthermore, we have assumed that $C(R) \equiv \widehat{C}_4$ also depends only on $r = R(t)$. The kinetic energy associated with the rotation of the brane is $\dot{\Theta}^2 = s_{IJ}\dot{\Theta}^I\dot{\Theta}^J$. From the form of the Lagrangian it is clear that there is a conserved energy as well as a conserved angular momentum around the 5-sphere, according to Noether's theorem.

The canonical momenta are

$$\begin{aligned}P &= \frac{\partial\mathcal{L}}{\partial\dot{R}} = \frac{B\dot{R}}{\sqrt{A - B\dot{R}^2 - D\dot{\Theta}^2}}, \\ P_I &= \frac{\partial\mathcal{L}}{\partial\dot{\Theta}^I} = \frac{Ds_{IJ}\dot{\Theta}^J}{\sqrt{A - B\dot{R}^2 - D\dot{\Theta}^2}}.\end{aligned} \tag{4.7}$$

The Hamiltonian for the probe brane is found by the Legendre transformation

$$\mathcal{H} = P\dot{R} + P_I\dot{\Theta}^I - \mathcal{L}, \tag{4.8}$$

and the conserved energy, $E = \mathcal{H}$, is

$$E = E(\dot{R}, \dot{\Theta}) = \frac{A}{\sqrt{A - B\dot{R}^2 - D\dot{\Theta}^2}} + C. \tag{4.9}$$

The conserved norm squared of the angular momentum is

$$J^2 = J^2(\dot{R}, \dot{\Theta}) = s^{IJ}P_IP_J = \frac{D^2\dot{\Theta}^2}{A - B\dot{R}^2 - D\dot{\Theta}^2}, \tag{4.10}$$

where s^{IJ} is the inverse of s_{IJ}, and where we have used that $s^{IJ}s_{IK}s_{JL}\dot{\Theta}^K\dot{\Theta}^L = \dot{\Theta}^2$. Eqs. (4.9) and (4.10) can now be inverted to yield $\dot{R}$ and $\dot{\Theta}$ as functions of E and J. First, from Eq. (4.10) one finds

$$\dot{\Theta}^2 = \frac{(A - B\dot{R}^2)J^2}{D(D + J^2)}. \tag{4.11}$$

Inserting this expression into Eq. (4.9) and solving for $\dot{R}^2$, one obtains the equations of motion for the probe brane

$$\dot{R}^2 = \frac{A}{B}\left[1 - \frac{A}{(E - C)^2}\frac{(D + J^2)}{D}\right], \tag{4.12}$$

$$\dot{\Theta}^2 = \frac{A^2}{(E - C)^2}\frac{J^2}{D^2}. \tag{4.13}$$

Here, E and J are considered as parameters of the brane trajectory, and A, B, C, D are quantities describing the background geometry as well as the 4-form field in the bulk. Since they depend just on the radial coordinate $r = R(t)$, the problem is analogous to that of a test particle in a central potential. Eq. (4.12) gives the radial velocity of the brane as a function of bulk time t. In principle, it can be integrated to give the radial trajectory $r = R(t)$ of the brane through the bulk.

4.3 Friedmann equation

The induced metric on the brane (4.4) implicitly depends on time. The function $g(R(t))$, multiplying the spatial part of the metric, plays the role of the scale factor, and hence the motion will induce an expansion or contraction on the brane. Let us write

$$\begin{aligned} \mathrm{d}s_4^2 &= \left(-f(R) + h(R)\dot{R}^2 + s(R)\dot{\Theta}^2\right)\mathrm{d}t^2 + g(R)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \\ &\equiv -\mathrm{d}\tau^2 + a^2(\tau)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \end{aligned} \tag{4.14}$$

where cosmic time τ on the brane is defined by

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = -f(R) + h(R)\dot{R}^2 + s(R)\dot{\Theta}^2 = \frac{f^2 g^3 \mathrm{e}^{-2\Phi}}{(E-C)^2}. \tag{4.15}$$

The second equality follows by substitution with Eqs. (4.12) and (4.13). By calculating the derivative of $a^2(\tau) = g(R(t(\tau)))$ with respect to τ, one obtains the relation

$$\left(\frac{a_\tau}{a}\right)^2 = \frac{1}{4}\left(\frac{g'}{g}\right)^2 \dot{R}^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2, \tag{4.16}$$

where a_τ is the abbreviation for $\frac{\mathrm{d}a}{\mathrm{d}\tau}$. Using Eqs. (4.15) and (4.12), one gets finally

$$\begin{aligned}\left(\frac{a_\tau}{a}\right)^2 &= \frac{1}{4}\left(\frac{g'}{g}\right)^2 \left[\frac{(E-C)^2 s\,\mathrm{e}^{2\Phi} - f(g^3 s + J^2 \mathrm{e}^{2\Phi})}{f\, g^3\, h\, s}\right] \\ &\equiv \frac{8\pi G_4}{3}\rho_{\mathrm{eff}}.\end{aligned} \tag{4.17}$$

In the second line, we have reinterpreted the cosmological evolution in terms of a mirage energy density ρ_{eff}, which is driving the expansion. However, ρ_{eff} does not correspond to any real matter on the brane. If there is no motion, $g' = \mathrm{d}g/\mathrm{d}R = 0$, then also $\rho_{\mathrm{eff}} = 0$.

This Friedmann equation is valid for any background of the form (4.1). In the special case of AdS$_5$-S$\times$S^5, the metric is given by (3.61), and the solution[1] for the 4-form field is $C = \widehat{C}_4 = -\frac{r^4}{L^4} + \frac{r_H^4}{2L^4}$. Then the Friedmann equation (4.17) becomes

$$\left(\frac{a_\tau}{a}\right)^2 = \frac{1}{L^2}\left[\left(2\tilde{E} + \frac{r_H^4}{L^4}\right)\frac{1}{a^4} - \frac{J^2}{L^2}\frac{1}{a^6} + \tilde{E}^2\frac{1}{a^8} + \frac{r_H^4 J^2}{L^6}\frac{1}{a^{10}}\right], \tag{4.18}$$

where we have absorbed the constant part of C in a redefinition of the energy: $\tilde{E} \equiv E - \frac{r_H^4}{2L^4}$. Since $g(R) = R^2/L^2 = a^2$, a cosmological expansion is induced if the brane moves towards bigger r and vice versa.

Each term in Eq. (4.18) can be associated with a different equation of state $P_{\mathrm{eff}} = \omega\rho_{\mathrm{eff}}$, namely

$$\frac{1}{a^4} \leftrightarrow \omega = \frac{1}{3}, \quad \frac{1}{a^6} \leftrightarrow \omega = 1, \quad \frac{1}{a^8} \leftrightarrow \omega = \frac{5}{3}, \quad \frac{1}{a^{10}} \leftrightarrow \omega = \frac{7}{3}. \tag{4.19}$$

The $1/a^4$ term dominates at late times or at big r, and reproduces the behavior of radiation. This term is always greater or equal than zero because $E \geq 0$. Further remarks on this point are made in paragraph 5.2.3. The early evolution is characterized by exotic matter with $\omega = 5/3$ and $\omega = 7/3$.

A deficiency is the absence of a term $1/a^3$, mimicking the evolution due to dust matter. On the other hand, one has to keep in mind, that Eq. (4.18) is valid

[1] For a derivation see [93] and Sec. 5.2.

only for an AdS_5-S×S^5 background. It is quite possible that other string theory backgrounds could lead to a more realistic Friedmann equation.

In the next chapter, we derive the Friedmann equation on a probe 3-brane in a 5-dimensional AdS_5-S bulk (see Eq. 5.29). Not surprisingly, it turns out that this is simply the above equation (4.18) with $J = 0$: by omitting the S^5 part, we have considered the brane to be located at a fixed point on the sphere, and so its angular momentum is zero. On the other hand, when there are more than one extra-dimension, there is greater freedom in the brane motion, and this leads to the additional terms in the Friedmann equation (4.18) with respect to the one in five dimensions.

4.4 Discussion

A general interesting feature of higher-dimensional theories is that the initial singularity problem can be naturally resolved. In Eq. (4.18), $a(\tau) = 0$ appears as a singularity to an observer on the 3-brane, whereas from the 10-dimensional perspective, $r = 0$ is a regular point corresponding to the horizon of the underlying black 3-brane solution. In this example, the initial singularity on the brane is an artefact because the embedding breaks down. More generally speaking, one expects that also general relativity is only an effective description of a more fundamental higher-dimensional theory, which should always be regular. Singularities such as the big bang then correspond to points, where the effective description breaks down.

An advantage of mirage cosmology is certainly its relative simplicity, and the fact that the formalism can be applied to arbitrary p-branes in a D-dimensional space-time. A major shortcoming, however, seems to be the lack of back-reactions or self-gravity. We are going to comment on this point at the end of Sec. 6.3.

In Chap. 6 and the following, we investigate other brane world models, which accommodate the back-reaction via junction conditions linking the real energy content of the brane to the geometry of the bulk. The disadvantage there is that the approach is limited to the case of one co-dimensison.

It not clear whether the exotic matter contributions (4.19) are compatible with experiments. Nucleosynthesis constraints suggest that the expansion of the universe is driven by real matter, and that it is not due to motion only. On the other hand, real matter *can* be included in this picture, for instance via the gauge field $F_{\mu\nu}$ in the DBI action (see Ref. [90]). The most general situation is obviously a combination of both. It is worthwhile investigating the degree to which the mirage matter contribution is important.

In the subsequent article, we are studying the evolution of perturbations on a probe brane moving through an AdS_5-S space-time.

Chapter 5

Perturbations on a moving D3-brane and mirage cosmology (article)

This chapter consists of the article 'Perturbations on a moving D3-brane and mirage cosmology', published in Phys.Rev.D66 (2002), see Ref. [22].

It is also available under http://lanl.arXiv.org/abs/hep-th/0206147.

Perturbations on a moving D3-brane and mirage cosmology

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We study the evolution of perturbations on a moving probe D3-brane coupled to a 4-form field in an AdS$_5$-Schwarzschild bulk. The unperturbed dynamics are parameterized by a conserved energy E and lead to a Friedmann-Robertson-Walker 'mirage' cosmology on the brane with a scale factor $a(\tau)$. The fluctuations about the unperturbed worldsheet are then described by a scalar field $\phi(\tau, \vec{x})$. We derive an equation of motion for ϕ, and find that in certain regimes of a the effective mass squared is negative. On an expanding BPS brane with $E = 0$ superhorizon modes grow as a^4 while subhorizon modes are stable. When the brane contracts, all modes grow. We also briefly discuss the case when $E > 0$, BPS anti-branes as well as non BPS branes. Finally, the perturbed brane embedding gives rise to scalar perturbations in the FRW universe. We show that ϕ is proportional to the gauge invariant Bardeen potentials on the brane.

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5.1 Introduction

The idea that our universe may be a 3-brane embedded in a higher dimensional space-time is strongly motivated by string and M theory, and it has recently received a great deal of attention. Much work has focused on the case in which the universe 3-brane is of co-dimension one [136, 127, 128] and the resulting cosmology (see *e.g.* [19, 43, 44]) and cosmological perturbation theory (*e.g.* [156, 115, 100, 129, 33, 96, 97]) have been studied in depth. When there is more than one extra dimension the Israel junction conditions, which are central to the 5-dimensional studies, do not apply and other approaches must be used [35, 90, 6]. In the 'mirage' cosmology approach [90, 92] the bulk is taken to be a given supergravity solution, and our universe is a *test* D3-brane which moves in this background space-time so that its back-reaction onto the bulk is neglected. If the bulk metric has certain symmetry properties, the unperturbed brane motion leads to FRW cosmology with a scale factor $a(\tau)$ on the brane [90, 144]. Our aim in this paper is to study the evolution of perturbations on such a moving brane. Given the probe nature of the brane, this question has many similarities with the study of the dynamics and perturbations of topological defects such as cosmic strings [76, 64, 72, 16].

Though we derive the perturbation equations in a more general case, we consider in the end a bulk with AdS_5-Schwarzschild$\times\mathrm{S}^5$ geometry which is the near horizon limit of the 10-dimensional black D3-brane solution. In this limit (using the AdS-CFT correspondence) black-hole thermodynamics can be studied via the probe D3-brane dynamics [36, 139]. As discussed in paragraph 5.2.1, we make the assumption that the D3-brane has no dynamics around the S^5 so that the bulk geometry is effectively AdS_5-Schwarzschild. Because of the generalized Birkhoff theorem [26], this 5-dimensional geometry plays an important rôle in work on co-dimension one brane cosmology. Hence links can be made between the unperturbed probe brane FRW cosmology discussed here and exact brane cosmology based on the junction conditions [144]. Similarly, the perturbation theory we study here is just one limit of the full, self-interacting and non Z_2-symmetric brane perturbation theory which has been studied elsewhere [129]. Comments will be made in the conclusions regarding generalizations of this work to the full 10-dimensional case.

Regarding the universe brane, the zeroth order (or background) solution is taken to be an infinitely straight brane whose motion is now constrained to be along the single extra dimension labelled by coordinate r. The brane motion is parameterized by a conserved positive energy E [90]. In AdS_5-Schwarzschild geometry and to an observer on the brane, the motion appears to be FRW expansion or contraction with a scale factor given by $a \propto r$. Both the perturbed and unperturbed brane dynamics will be obtained from the Dirac-Born-Infeld action for type IIB superstring theory (see *e.g.* [14]),

$$S_{D3} = -T_3 \int \mathrm{d}^4\sigma \sqrt{-\det(g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} + \widehat{B}_{\mu\nu})} - \rho_3 \int \mathrm{d}^4\sigma\, \widehat{C}_4. \tag{5.1}$$

Here σ^μ $(\mu = 0, 1, 2, 3)$ are coordinates on the brane world-sheet, T_3 is the brane tension, and in the second (Wess-Zumino) term ρ_3 is the brane charge under a Ramond-Ramond 4-form field living in the bulk. We will write

$$\rho_3 = qT_3, \tag{5.2}$$

so that $q = (-)1$ for BPS (anti-)branes. In Eq. (5.1) $g_{\mu\nu}$ is the induced metric and $F_{\mu\nu}$ the field strength tensor of gauge fields on the brane. The quantities $\widehat{B}_{\mu\nu}$ and $\widehat{C}_4$ are the pull-backs of the Neveu-Schwarz (NS) 2-form, and the Ramond-Ramond (RR) 4-form field in the bulk[1]. In the background we consider, the dilaton is a constant and we set it to zero. In general the brane will not move slowly, and hence the square root in the DBI part of (5.1) may not be expanded: we will

[1]If we want to stress that a certain quantity is the pull-back of a bulk tensor, we use the widehat symbol. However, for simplicity, we denote the induced metric by $g_{\mu\nu}$ rather than $\widehat{g}_{\mu\nu}$

consider the full non linear action. Finally, notice that since the 4-dimensional Riemann scalar does not appear in (5.1) (and it is not inherited from the background in this probe brane approach) there is no brane self-gravity. Hence the 'mirage' cosmology we discuss here is solely sourced by the brane motion, and it leads to effects which are not present in 4-dimensional Einstein gravity. The lack of brane self-gravity is a serious limitation. However, in certain cases it may be included, for instance by compactifying the background space-time as discussed in [30] (see also [35]). Generally this leads to bi-metric theories. Even in that case, the mirage cosmology scale factor $a(\tau)$ which we discuss below plays an important rôle and hence we believe it is of interest to study perturbations in this 'probe brane' approach.

Deviations from the infinitely straight moving brane give rise to perturbations around the FRW solution. Are these 'wiggles' stretched away by the expansion, or on the contrary do they grow, leading to instabilities? To answer this question, we exploit the similarity with topological defects and make use of the work developed in that context by Garriga and Vilenkin [64], Guven [72], and Battye and Carter [16]. The perturbation dynamics are studied through a scalar field $\phi(\sigma)$ whose equation of motion is derived from the action (5.1). We find that, for an observer comoving with the brane, ϕ has a tachyonic mass in certain ranges of r which depend on the conserved energy E characterizing the unperturbed brane dynamics. We discuss the evolution of the modes ϕ_k for different E and show that in many cases the brane is unstable. In particular, both sub and superhorizon modes grow for a brane falling into the black-hole. It remains an open question to see if brane self-gravity, neglected in this approach, can stabilize the system.

Finally, we also relate ϕ to the standard 4-dimensional gauge invariant scalar Bardeen potentials Φ and Ψ on the brane. We find that $\Phi \propto \Psi \propto \phi$ (no derivatives of ϕ enter into the Bardeen potentials).

The work presented here has some overlap with that of Carter *et al.* [41] who also studied perturbations on moving charged branes in the limit of negligible self-gravity. Their emphasis was on trying to mimic gravity on the brane, and in addition they included matter on the brane. Here we consider the simplest case in which there is no matter on the brane, namely $F_{\mu\nu} = 0$ in Eq. (5.1). Our focus is on studying the evolution of perturbations solely due to motion of the brane: we expect the contribution of these perturbations to be important also when matter is included. Moreover, we hope that this study may more generally be of interest for the dynamics and perturbations of moving D-branes in non BPS backgrounds.

The outline of the paper is as follows. In section 5.2 we link our 5-dimensional metric to the 10-dimensional black D3-brane solution and specify the unperturbed embedding of the probe brane. To determine its dynamics from the action (5.1) the bulk 4-form RR field must be specified. We discuss the normalization of this field. At the end of the section we summarize the motion of the probe brane by means of an effective potential. Comments are made regarding the Friedmann equation for an observer on the brane. In section 5.3 we consider small deviations from the background brane trajectory and investigate their evolution.

The equation of motion for ϕ is derived, and we solve it in various regimes, commenting on the resulting instabilities. In section 5.4 we link ϕ to the scalar Bardeen potentials on the brane. Finally, in section 5.5 we summarize our results.

5.2 Unperturbed dynamics of the D3-brane

In this section we discuss the background metric, briefly review the unperturbed D3-brane dynamics, and comment on the cosmology as seen by an observer on the brane. The reader is referred to [90, 122] for a more detailed analysis on which part of this section is based.

5.2.1 Background metric and brane scale factor

For the reasons mentioned in the introduction, we focus mainly on an AdS$_5$-S$\times$S^5 bulk space-time. This is closely linked to the 10-dimensional black 3-brane supergravity solution [78, 2, 93] which describes N coincident D3-branes carrying RR charge $Q = NT_3$ and which is given by

$$\mathrm{d}s_{10}^2 = H_3^{-1/2}(-F\mathrm{d}t^2 + \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}) + H_3^{1/2}\left(\frac{\mathrm{d}r^2}{F} + r^2\mathrm{d}\Omega_5^2\right), \tag{5.3}$$

where the coordinates $(t, \vec{x})$ are parallel to the N D3-branes, $\mathrm{d}\Omega_5^2$ is the line element on a 5-sphere, and

$$H_3(r) = 1 + \frac{L^4}{r^4}, \qquad F = 1 - \frac{r_H^4}{r^4}. \tag{5.4}$$

The quantity L is the AdS$_5$ curvature radius and the horizon r_H vanishes when the ADM mass equals Q. The link between the metric parameters L, r_H and the string parameters N, T_3 is given *e.g.* in [93]. The corresponding bulk RR field may also be found in [93].

The near horizon limit of the metric (5.3) is AdS$_5$-S$\times$S^5 space-time [2]. Our universe is taken to be a D3-brane moving in this background. We make the following two assumptions: first, the universe brane is a 'probe' so that its back-reaction on the bulk geometry is neglected. This may be justified if $N \gg 1$. Second, the probe is assumed to have no dynamics around S^5 so that it is constrained to move only along the radial direction r. This is a consistent solution of the *unperturbed* dynamics since the brane has a conserved angular momentum about the S^5, and this may be set to zero [90, 144]. In section 5.3 we assume that is also true for the perturbed dynamics. Thus in the remainder of this paper we consider an AdS$_5$-S bulk space-time with metric

$$\begin{aligned}\mathrm{d}s_5^2 &= -f(r)\mathrm{d}t^2 + g(r)\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} + h(r)\mathrm{d}r^2 \\ &\equiv G_{MN}\mathrm{d}x^M\mathrm{d}x^N,\end{aligned} \tag{5.5}$$

where (for $r > r_H$)

$$f(r) = \frac{r^2}{L^2}\left(1 - \frac{r_H^4}{r^4}\right), \qquad g(r) = \frac{r^2}{L^2}, \qquad h(r) = \frac{1}{f(r)}. \tag{5.6}$$

In the limit $r_H \to 0$ this becomes pure AdS_5.

More generally, by symmetry, a stack of non rotating D3-branes generates a metric of the form $\mathrm{d}s_{10}^2 = \mathrm{d}s_5^2 + k(r)\mathrm{d}\Omega_5^2$, where $\mathrm{d}s_5^2$ is given in Eq. (5.5) [145]. In this case, since the metric coefficients are independent of the angular coordinates $(\theta^1, \cdots, \theta^5)$, the unperturbed brane dynamics are always characterized by a conserved angular momentum around the S^5 [90]. As a result of the second assumption above, we are thus effectively led to consider metrics of the form (5.5). Hence, for the derivation of both the unperturbed and perturbed equations of motion, we keep f, g, h arbitrary and consider the specific form (5.6) only at the end.

The embedding of the probe D3-brane is given by $x^M = X^M(x^\mu)$. (We have used reparametrization invariance to choose the intrinsic worldsheet coordinates $\sigma^\mu = x^\mu$.) For the unperturbed trajectory we consider an infinitely straight brane parallel to the x^μ hyperplane but free to move along the r direction:

$$X^\mu = x^\mu, \qquad X^4 = R(t). \tag{5.7}$$

Later, in section 5.3, we will consider a perturbed brane for which $X^4 = R(t) + \delta R(t, \vec{x})$.

The induced metric on the brane is given by

$$g_{\mu\nu} = G_{MN}(X)\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}, \tag{5.8}$$

so that the line element on the unperturbed brane worldsheet is

$$\mathrm{d}s_4^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(f(R) - h(R)\dot{R}^2)\mathrm{d}t^2 + g(R)\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} \equiv -\mathrm{d}\tau^2 + a^2(\tau)\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}. \tag{5.9}$$

An observer on the brane therefore sees a homogeneous and isotropic universe in which the time τ and the scale factor $a(\tau)$ are given by

$$\tau = \int \sqrt{(f - h\dot{R}^2)}\mathrm{d}t, \qquad a(\tau) = \sqrt{g(R(\tau))}. \tag{5.10}$$

The properties of the resulting Friedmann equation depend on $f(R), g(R), h(R)$ (*i.e.* the bulk geometry) as well as $\dot{R}$ (the brane dynamics) as discussed in [90, 144] and summarized briefly below.

5.2.2 Brane action and bulk 4-form field

In AdS_5-S geometry, B_{MN} vanishes, and we do not consider the gauge field $F_{\mu\nu}$ on the brane. (For a detailed discussion of the unperturbed brane dynamics with and without $F_{\mu\nu}$, which essentially corresponds to radiation on the brane, see [90, 144]. Non zero B_{MN} has been discussed in [163].) Thus the brane action (5.1) reduces to

$$S_{D3} = -T_3 \int \mathrm{d}^4x\sqrt{-g} - \rho_3 \int \mathrm{d}^4x\,\widehat{C}_4, \tag{5.11}$$

where

$$g = \det(g_{\mu\nu}), \qquad \widehat{C}_4 = C_{MNSR}\frac{\partial X^M}{\partial x^0}\frac{\partial X^N}{\partial x^1}\frac{\partial X^S}{\partial x^2}\frac{\partial X^R}{\partial x^3}, \tag{5.12}$$

and C_{MNSR} are components of the bulk RR 4-form field. The first term in Eq. (5.11) is just the Nambu-Goto action.

In the gauge (5.7), g and $\widehat{C}_4$ depend on t only through R. Thus rather than varying (5.11) with respect to X^M and then integrating the equations of motion, it is more straightforward to obtain the equations of motion from the Lagrangian

$$\mathcal{L} = -\sqrt{-g} - C \;\; = -\sqrt{fg^3 - g^3h\dot{R}^2} - C, \tag{5.13}$$

where $C = C(R) = \frac{\rho_3}{T_3}\widehat{C}_4 = q\widehat{C}_4$. Since $\mathcal{L}$ does not explicitly depend on time, the brane dynamics are parameterized by a (positive) conserved energy $E = \frac{\partial\mathcal{L}}{\partial\dot{R}}\dot{R} - \mathcal{L}$ from which

$$\dot{R}^2 = \frac{f}{h}\left(1 - \frac{fg^3}{(E-C)^2}\right). \tag{5.14}$$

Transforming to brane time τ defined in equation (5.10) yields

$$R_\tau^2 = \frac{(E-C)^2}{fg^3h} - \frac{1}{h}, \tag{5.15}$$

where the subscript denotes a derivative with respect to τ.

In order to analyze the brane dynamics in AdS$_5$-S space-time, where f, g and h are given in Eq. (5.6), one must finally specify $C(R)$ or equivalently the 4-form potential C_{MNSR}. To that end[2] recall that the 5-dimensional bulk action is

$$S_5 = \frac{1}{2\kappa_5^2}\int \mathrm{d}^5x\sqrt{-G}(R - 2\Lambda) - \frac{1}{4\kappa_5^2}\int F_5 \wedge *F_5, \tag{5.16}$$

where Λ is the bulk cosmological constant[3] and $F_5 = \mathrm{d}C_4$ is the 5-form field

[2] For the 10-dimensional AdS$_5$-S$\times$S^5 geometry the solution for the 4-form field is given, for example, in [93]. For completeness, we rederive the result starting directly from the 5-dimensional metric (5.6).

[3] Note that there is no cosmological constant in the fundamental 10-dimensional supergravity action 2.1. Here, Λ appears because we consider only the AdS$_5$ part. It represents the negative cosmological constant of AdS$_5$ which, together with the positive Ricci curvature form the 5-sphere, adds up to zero. Thus Λ is needed for consistency of the equations of motion.

strength associated with the 4-form C_4. The resulting equations of motion are

$$R_{MN} = \frac{2}{3}\Lambda G_{MN} + \frac{1}{2\cdot 4!}\left(F_{MBCDE}F_N{}^{BCDE} - \frac{4}{3\cdot 5}F_{ABCDE}F^{ABCDE}G_{MN}\right), \tag{5.17}$$

$$\mathrm{d} * F_5 = \frac{1}{2}\frac{1}{\sqrt{fg^3h}}\left(\left(\frac{f'}{f} + 3\frac{g'}{g} + \frac{h'}{h}\right)F_{01234} - 2F'_{01234}\right)\mathrm{d}r = 0, \tag{5.18}$$

where the prime denotes a derivative with respect to r. In AdS$_5$-S, $R_{MN} = -\frac{4}{L^2}G_{MN}$ and Eq. (5.18) gives

$$\frac{L^3}{r^3}\left(\frac{3}{r}F_{01234} - F'_{01234}\right) = 0 \qquad \Longrightarrow \qquad F_{01234} = c\frac{r^3}{L^4}, \tag{5.19}$$

where c is a dimensionless constant (see for example [40]). (Note that this solution satisfies $\mathrm{d}F_5 = 0$ since the only non zero derivative is $\partial_4 F_{01234}$ which vanishes on antisymmetrizing.) Integration gives

$$C_{0123} = v\frac{r^4}{L^4} + w, \tag{5.20}$$

where $v = c/4$ and w are again dimensionless constants. Hence the function $C(r)$ appearing in Eq. (5.13) is

$$C(r) = qC_{0123} = qv\frac{r^4}{L^4} + qw. \tag{5.21}$$

In 10 dimensions the constant c (and hence v) is fixed by the condition $\int *F = Q$, and w may be determined by imposing (before taking the near horizon limit and hence with metric (5.3)) that the 4-form potential should die off at infinity [93]. This second argument is not applicable here. Instead, we fix v and w in the following way: consider the motion of the unperturbed brane seen by a bulk observer with time coordinate t. One can define an effective potential V^t_{eff} through

$$\frac{1}{2}\dot{R}^2 + V^t_{\text{eff}} \equiv E \tag{5.22}$$

so that on using equation (5.14),

$$V^t_{\text{eff}}(E,q,R) = E - \frac{1}{2}\left(\frac{R}{L}\right)^4\alpha^2\left[1 - \left(\frac{R}{L}\right)^8\frac{\alpha}{(E-C)^2}\right] \tag{5.23}$$

(see Fig. 5.2) where

$$\alpha = 1 - \frac{r_H^4}{R^4},$$

and $C = C(R)$ is given in Eq. (5.21). We now use the fact that there is no net force between static BPS objects of like charge, and hence in this case the effective potential should be identically zero. Here, such a configuration is characterized by $r_H = 0, q = 1, E = 0$: imposing that $V^t_{\text{eff}} = 0$ for all R, forces $v = \pm 1$ and, in this limit, $w = 0$. Second, we normalize the potential such that $V^t_{\text{eff}}(E, q = 1, R \to \infty) = 0$ for arbitrary values of the energy E and r_H. This leads to

$$v = -1, \qquad w = +\frac{r_H^4}{2L^4}. \tag{5.24}$$

In particular for $E = 0$, the brane has zero kinetic energy at infinity. Even in this case the potential is not flat, unless $r_H = 0$, as can be see in Fig. 5.1. According to this normalization

$$C(r) = -q\frac{r^4}{L^4} + q\frac{r_H^4}{2L^4} < 0 \tag{5.25}$$

as in the 10-dimensional case [93]. Notice that, since the combination appearing in the equation of motion for R is $E - C$, the constant w only acts to shift the energy. For later purposes, we define the shifted energy $\tilde{E}$ by

$$\tilde{E} = E - qw = E - q\frac{r_H^4}{2L^4}. \tag{5.26}$$

Figure 5.1: $V^t_{\text{eff}}(E, q, R)$ for $E = 0$, $q = 1$, $L = 4$ and different values of r_H. For $R \to \infty$ the potential goes to zero according to our normalization. When $r_H = 0$, the potential is exactly flat.

Finally, we comment that substitution of Eq. (5.19) into Eq. (5.17) determines the bulk cosmological[4] constant to be given by $L^2\Lambda = -6 - c^2/4 = -10$.

[4]Equivalently we could have started from the 10-dimensional SUGRA action, used the 10-dimensional solution for F_5 (which is identical to (5.25)) and then integrated out over the 5-sphere. After definition of the 5-dimensional Newton constant in terms of the 10-dimensional one, the above cosmological constant term is indeed obtained, coming from the 5-sphere Ricci scalar.

5.2.3 Brane dynamics and Friedmann equation

We now make some comments regarding the unperturbed motion of the 3-brane through the bulk, $R(\tau)$, as seen by an observer on the brane. This will be useful in section 5.3 when discussing perturbations. Recall that since $a(\tau) = R(\tau)/L$ (see Eq. (5.10)), an 'outgoing' brane leads to cosmological expansion. Contraction occurs when the brane moves inward. For the observer on the brane, one may define an effective potential by

$$\frac{1}{2}{R_\tau}^2 + V_{\rm eff}^\tau = E, \tag{5.27}$$

whence, from Eq. (5.15),

$$V_{\rm eff}^\tau(E,q,R) = E + \frac{1}{2}\left(\frac{L}{R}\right)^6\left[\alpha\left(\frac{R}{L}\right)^8 - (E-C)^2\right]. \tag{5.28}$$

Consider a BPS brane $q = +1$ (see Fig. 5.3). As noted above, for $r_H = E = 0$ one has $V_{\rm eff}^\tau = 0$ so that the potential is flat. For $r_H \neq 0$, $V_{\rm eff}^\tau$ contains a term $\propto -R^{-6}$, and the probe brane accelerates toward the horizon, which is reached in finite (τ-)time. On the other hand, for a bulk observer with time t, it takes infinite time to reach the horizon where $V_{\rm eff}^t = E$ (see Fig. 5.2).

From Eqs. (5.15) and (5.21) it is straightforward to derive a Friedmann-like equation for the brane scale factor $a(\tau)$ [90, 144]:

$$H^2 = \left(\frac{a_\tau}{a}\right)^2 = \frac{1}{L^2}\left[\frac{\tilde{E}^2}{a^8} + \frac{1}{a^4}\left(2q\tilde{E} + \frac{r_H^4}{L^4}\right) + (q^2-1)\right]. \tag{5.29}$$

The term in $1/a^8$ (a 'dark fluid' with equation of state $P = 5/3\rho$) dominates at early times. The second term in $1/a^4$ is a 'dark radiation' term. As discussed in [144], the part proportional to r_H corresponds to the familiar dark radiation term in conventional Z_2-symmetric (junction condition) brane cosmology, where it is associated with the projected bulk Weyl tensor. When $\tilde{E}$ is non zero, Z_2 symmetry is broken[5] [144] and this leads to a further dark radiation term [40, 98]. The last term in Eq. (5.29) defines an effective 4-dimensional cosmological constant $\frac{\Lambda_4}{3} \equiv \frac{1}{L^2}(q^2-1)$ which vanishes if the (anti) brane is BPS (*i.e.* $q = \pm 1$). All these terms have previously been found in both mirage cosmology and conventional brane cosmology [144, 40].

Notice that the dark radiation term above has a coefficient

$$\mu \equiv 2q\tilde{E} + \frac{r_H^4}{L^4} = 2qE - \frac{r_H^4}{L^4}(q^2-1), \tag{5.30}$$

which is positive for $q = +1$ (since $E \geq 0$). However, for BPS anti-branes $q = -1$, the coefficient (5.30) is negative unless $E = 0$. Thus when $E \neq 0$ and $q = -1$

[5] When making the link between mirage cosmology and the junction condition approach, $\tilde{E} \propto M_- - M_+$ where $M_\pm$ are the black-hole masses on each side of the brane [144].

Figure 5.2: $V^t_{\text{eff}}(E,q,R)$ for $E=2$, $r_H=1$, $L=4$. For a BPS-brane ($q=1$), $V^t_{\text{eff}} \to 0$ as $R \to \infty$ according to our normalization. This should be contrasted with a non BPS brane *e.g.* with $q=1.2$. Note that $V^t_{\text{eff}}(E,q,R=r_H)=E$. Any inwardly moving (contracting) brane takes an infinite amount of t-time to reach the horizon.

there is a regime of R for which H^2 is negative. In Fig. 5.3 this is represented by the forbidden region where the potential exceeds the total energy E. At $V^\tau_{\text{eff}}=E$ the Hubble parameter is zero and an initially expanding brane starts contracting. On the contrary, we do not obtain bouncing solutions in our setup, regardless of the values of q and E. Bouncing and oscillatory universes are discussed in *e.g.* [112, 84, 31].

The Friedmann equation (5.29) can be solved exactly. In the BPS case, $\Lambda_4=0$, the solution is

$$a(\tau)^4 = a_i^4 + \frac{4\mu}{L^2}(\tau-\tau_i)^2 \pm \frac{4}{L}(\tau-\tau_i)(\tilde{E}^2+\mu a_i^4)^{1/2}, \tag{5.31}$$

where a_i is the value of the scale factor at the initial time τ_i, and the $\pm$ determines whether the brane is moving radially inward or outward. In the next section when we solve the perturbation equations, it will be sufficient to consider regimes in which only one of the terms in Eq. (5.29) dominates. These will be given in section 5.3.

One might wonder whether it is possible to obtain a term $\propto 1/a^3$ (dust) in the Friedmann equation, and also one corresponding to physical radiation on the brane (rather than dark radiation). Physical radiation comes from taking $F_{\mu\nu} \neq 0$ in (5.1) [90], and a 'dark' dust term has been obtained in the non BPS background studied in [30]. Finally, a curvature term $1/a^2$ has been obtained in [164].

Figure 5.3: $V_{\text{eff}}^{\tau}(E,q,R)$ for the same parameters as in Fig. 5.2. A BPS brane has zero kinetic energy at infinity corresponding to a vanishing cosmological constant on the brane. Otherwise, the cosmological constant is $\propto q^2-1$. A BPS anti-brane is allowed to move only in a restricted range of R: after having reached a maximal scale factor, the universe starts contracting. Any inwardly moving brane falls into the black-hole in a finite τ.

5.3 Perturbed equations of motion

In this section we consider perturbations of the brane position about the zeroth order solution $R(t)$ given in Eq. (5.14). Once again we work with the metric (5.5), specializing to AdS$_5$-S geometry only at the end. The perturbed brane embedding $X^4 = R(t) + \delta R(t,\vec{x})$ leads to perturbations $\delta g_{\mu\nu}$ of the induced metric on the brane and these are discussed in section 5.4. Note that these perturbations about the flat homogenous and isotropic solution are not sourced by matter on the brane, and their evolution will depend on the unperturbed brane dynamics and hence on E. We now derive an equation for the evolution of the perturbed brane to see if there are instabilities in the system.

5.3.1 The second order action

Since we consider a co-dimension one brane, the fluctuations about the unperturbed moving brane can be described by a single scalar field $\phi(x^\mu)$ living on the unperturbed brane world sheet [72]. To describe the dynamics of $\phi(x^\mu)$ (which is defined below), we use the covariant formalism developed in [72] to study perturbed Nambu-Goto walls. (For other applications, see also [64, 34].)

The perturbed brane embedding is given by

$$\tilde{X}^M(t,\vec{x}) = X^M(t) + \phi(t,\vec{x}) n^M(t), \tag{5.32}$$

where $X^M(t)$ is the unperturbed embedding, and physical perturbations are only those transverse to the brane (see also section 5.4). The unit space-like normal to the unperturbed brane, $n^M(t) = n^M(X(t))$, is defined through

$$G_{MN} n^M \frac{\partial X^N}{\partial x^\mu} = 0, \qquad G_{MN} n^M n^N = 1, \tag{5.33}$$

so that

$$n^M = \left(\dot{R}\sqrt{\frac{h}{f(f-h\dot{R}^2)}}, 0, 0, 0, \sqrt{\frac{f}{h(f-h\dot{R}^2)}} \right). \tag{5.34}$$

Thus for a 5-dimensional observer comoving with the brane, ϕ (which has dimensions of length) is the measured deviation from the background solution of the previous section [64]. For an observer living on the brane, the perturbations in the FRW metric generated by ϕ are discussed in section 5.4 in terms of the gauge invariant scalar Bardeen potentials.

An equation of motion for ϕ can be obtained by substituting (5.32) into the action (5.11) and expanding to second order in ϕ. The terms linear in ϕ give the background (unperturbed) equations of motion studied in the previous section; now we are interested in the terms quadratic in ϕ which give the linearized equations of motion. A similar analysis was carried out by Garriga and Vilenkin [64] for Nambu-Goto cosmic domain walls in Minkowski space and was generalized by Guven [72] for arbitrary backgrounds. For the action (5.11), the quadratic term is [34]

$$S_{\phi^2} = -\frac{1}{2}\int d^4x\sqrt{-g}\left[(\widehat{\nabla}_\mu\phi)(\widehat{\nabla}^\mu\phi) - \left(k^\mu{}_\nu k^\nu{}_\mu + R_{MN}n^M n^N\right)\phi^2 \right]. \tag{5.35}$$

Here $\widehat{\nabla}$ is the covariant derivative with respect to the induced metric $g_{\mu\nu}$, and the extrinsic curvature tensor $k_{\mu\nu}$ on the brane is given by

$$k_{\mu\nu} = -(\nabla_N n_M)\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}, \tag{5.36}$$

where ∇ is the covariant derivative with respect to the 5-dimensional metric G_{MN}. Finally, R_{MN} is the Ricci tensor of the metric G_{MN}. Apart from ϕ, all the terms in (5.35) are unperturbed quantities. Note that there is no contribution to S_{ϕ^2} from the Wess-Zumino term of the action (5.11): all terms quadratic in ϕ cancel since C_{0123} is the only non zero component of the 4-form field. However, C does enter into the term linear in ϕ and hence into the background equations of motion, as analyzed in the previous sections.

Variation of the action (5.35) with respect to ϕ leads to the equation of motion

$$\widehat{\nabla}^\mu\widehat{\nabla}_\mu\phi + \left(k^\mu{}_\nu k^\nu{}_\mu + R_{MN}n^M n^N\right)\phi = 0 \tag{5.37}$$

or equivalently

$$\widehat{\nabla}^{\mu}\widehat{\nabla}_{\mu}\phi - m^2\phi = 0, \tag{5.38}$$

where

$$m^2 = -\left(k^{\mu}{}_{\nu}k^{\nu}{}_{\mu} + R_{MN}n^M n^N\right). \tag{5.39}$$

To determine the extrinsic curvature contribution to (5.39), it is simpler to calculate first the 5-dimensional extrinsic tensor or second fundamental form defined by Eq. (3.22)

$$K_{MN} = -q^{B}{}_{N}q^{C}{}_{M}\nabla_C n_B, \tag{5.40}$$

where $q^{B}{}_{N}$ acts as projection tensor onto the brane. It is the mixed tensor associated with the first fundamental form

$$q_{MN} = G_{MN} - n_M n_N. \tag{5.41}$$

We then use the fact that

$$k^{\mu}{}_{\nu}k^{\nu}{}_{\mu} = K^{M}{}_{N}K^{N}{}_{M}. \tag{5.42}$$

On defining T by

$$T \equiv \left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = f - h\dot{R}^2 = \frac{f^2 g^3}{(E-C)^2}, \tag{5.43}$$

the non zero components of $K^{M}{}_{N}$ are

$$\begin{aligned}
K^{0}{}_{0} &= -\frac{1}{T^{5/2}}f^{3/2}h^{1/2}\left(\ddot{R} - \frac{f'}{f}\dot{R}^2 + \frac{1}{2}\frac{h'}{h}\dot{R}^2 + \frac{1}{2}\frac{f'}{h}\right), && (5.44)\\
K^{0}{}_{4} &= \frac{h\dot{R}}{f}K^{0}{}_{0}, && (5.45)\\
K^{1}{}_{1} &= -\frac{1}{T^{1/2}}\left(\frac{f}{h}\right)^{1/2}\frac{1}{2}\frac{g'}{g} = K^{2}{}_{2} = K^{3}{}_{3}, && (5.46)\\
K^{4}{}_{4} &= \frac{h\dot{R}^2}{f}K^{0}{}_{0} && (5.47)
\end{aligned}$$

so that

$$k^{\mu}{}_{\nu}k^{\nu}{}_{\mu} = K^{M}{}_{N}K^{N}{}_{M} = \frac{1}{T}\frac{f}{h}\left[3\left(\frac{g'}{g}\right)^2 + 3\frac{g'}{g}\frac{C'}{E-C} + \left(\frac{C'}{E-C}\right)^2\right]. \tag{5.48}$$

The Ricci term is

$$\begin{aligned}
R_{MN}n^M n^N = &-\frac{1}{4h}\left[2\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 + 3\frac{f'}{f}\frac{g'}{g} - \frac{f'}{f}\frac{h'}{h}\right]\\
&+ \frac{3}{4}\frac{1}{T}\frac{f}{h}\left[\frac{f'}{f}\frac{g'}{g} - 2\frac{g''}{g} + \left(\frac{g'}{g}\right)^2 + \frac{g'}{g}\frac{h'}{h}\right].
\end{aligned} \tag{5.49}$$

Collecting these results gives

$$\begin{aligned} m^2 = &-\frac{3}{4}\frac{(E-C)^2}{fg^3h}\left[\frac{f'}{f}\frac{g'}{g} - 2\frac{g''}{g} + 5\left(\frac{g'}{g}\right)^2 + \frac{g'}{g}\frac{h'}{h} + 4\frac{g'}{g}\frac{C'}{E-C}\right. \\ &\left. + \frac{4}{3}\left(\frac{C'}{E-C}\right)^2\right] \\ &+ \frac{1}{4h}\left[2\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 + 3\frac{f'}{f}\frac{g'}{g} - \frac{f'}{f}\frac{h'}{h}\right]. \end{aligned} \tag{5.50}$$

In the remainder of this section we try to obtain approximate solutions for ϕ from equation (5.38). Some aspects of this calculation are clearer in brane time τ, and others in conformal time η (where $\eta = \int \mathrm{d}\tau/a(\tau)$). Of course the results are independent of the coordinate system. For these reasons we have decided to present both approaches, beginning with brane time.

5.3.2 Evolution of perturbations in brane time τ

On using the definition of brane time τ in equation (5.10), the kinetic term in (5.37) is given by

$$\widehat{\nabla}^\mu \widehat{\nabla}_\mu \phi = -\phi_{\tau\tau} - 3H\phi_\tau + \frac{1}{a^2}\left(\phi_{x^1x^1} + \phi_{x^2x^2} + \phi_{x^3x^3}\right). \tag{5.51}$$

(In conformal time the factor of a^{-2} multiplying the spatial derivatives disappears, see below.) We now change variables to $\varphi = a^{3/2}\phi$ so that Eq. (5.38) becomes

$$\varphi_{\tau\tau} - \frac{1}{a^2}\left(\varphi_{x^1x^1} + \varphi_{x^2x^2} + \varphi_{x^3x^3}\right) + M^2(\tau)\varphi = 0, \tag{5.52}$$

where

$$\begin{aligned} M^2(\tau) &= m^2 - \frac{3}{4}\left[\left(\frac{a_\tau}{a}\right)^2 + 2\frac{a_{\tau\tau}}{a}\right] \\ &= m^2 - \frac{3}{4}\left[\frac{g''}{g}R_\tau^2 - \frac{1}{4}\left(\frac{g'}{g}\right)^2 R_\tau^2 + \frac{g'}{g}R_{\tau\tau}\right] \\ &= \frac{3}{4}\frac{(E-C)^2}{fg^3h}\left[-\frac{1}{2}\frac{f'}{f}\frac{g'}{g} + \frac{g''}{g} - \frac{13}{4}\left(\frac{g'}{g}\right)^2 - \frac{1}{2}\frac{g'}{g}\frac{h'}{h} - 3\frac{g'}{g}\frac{C'}{E-C}\right. \\ &\quad\left. - \frac{4}{3}\left(\frac{C'}{E-C}\right)^2\right] \\ &\quad + \frac{1}{4h}\left[2\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 + 3\frac{f'}{f}\frac{g'}{g} - \frac{f'}{f}\frac{h'}{h} + 3\frac{g''}{g} - \frac{3}{4}\left(\frac{g'}{g}\right)^2 - \frac{3}{2}\frac{g'}{g}\frac{h'}{h}\right]. \end{aligned} \tag{5.53}$$

This expression is valid for any f, g, and h. We now specialize to AdS$_5$-S geometry in which case

$$\begin{aligned} M^2(\tau) &= \frac{1}{L^2}\left[-\frac{33}{4}\frac{\tilde{E}^2}{a^8} + \frac{3}{4}\frac{1}{a^4}\left(2q\tilde{E} + \frac{r_H^4}{L^4}\right) - \frac{25}{4}(q^2-1)\right] \\ &= -\frac{33}{4}H^2 + \frac{9}{a^4L^2}\left(2q\tilde{E} + \frac{r_H^4}{L^4}\right) + 2\frac{q^2-1}{L^2}. \end{aligned} \tag{5.54}$$

Notice that there are regimes of a in which $M^2 < 0$ such as, for instance, for small a where the a^{-8} term dominates, and furthermore that the location of these regimes depends on the energy E of the brane. We also see that since $M^2 \sim H^2$, instabilities will occur for modes with a wavelength greater than H^{-1}. Figure 5.4 shows the typical shape of M^2 as a function of a for fixed energy and different q. In the following, we discuss only cases with $q^2 \geq 1$ as then the 4-dimensional cosmological constant is positive.

Figure 5.4: The dimensionless quantity M^2L^2 (on the vertical axis) is plotted as a function of a for $E = 1, L = 1, r_H = 1$. Here, the effective mass squared is positive in a certain range only for the BPS-brane. Note that the negative M^2L^2 region is not hidden behind the horizon.

Analysis of equation (5.52) is simpler in Fourier space where

$$\varphi_k(\tau) = \int \mathrm{d}^3x\, \varphi(\tau, \vec{x})\mathrm{e}^{-i\vec{k}\cdot\vec{x}}, \tag{5.55}$$

and k is a comoving wave number related to the physical wave number k_p by $k = ak_p$. Thus (5.52) becomes

$$\varphi_{k,\tau\tau} + \frac{1}{a^2}\left(k^2 - k_c^2(\tau)\right)\varphi_k = 0, \tag{5.56}$$

where the time dependent critical wave number $k_c^2(\tau)$ is given by

$$k_c^2(\tau) = -M^2(\tau)a^2. \tag{5.57}$$

One might suppose that for $M^2 > 0$ all modes are stable. However, due to the τ-dependence of k_c this is not necessarily true (as we shall see in equation (5.71)).

Our aim now is to determine the a-dependence of φ_k. We proceed in the following way: notice first that the Friedmann equation (5.29) and the expression for $M^2(\tau)$ in (5.54) both contain terms in a^{-8}, a^{-4}, and a^0. We will focus on a regime in which one of these terms dominates. Then the Friedmann equation can be solved for $a(\tau)$ which, on substitution into (5.54), gives $M^2(\tau)$. A final substitution of $M^2(\tau)$ into the perturbation equation (5.56) for φ_k enables this equation to be solved in each regime. We consider the following cases: *i)* $q = +1$, *ii)* $q = -1$ and *iii)* $q^2 > 1$.

BPS brane: $q = +1$

For a BPS brane, the Friedmann equation (5.29) and effective mass $M^2(\tau)$ are given by

$$H^2 = \frac{1}{L^2}\left[\frac{\tilde{E}^2}{a^8} + \frac{2E}{a^4}\right], \tag{5.58}$$

$$M^2(\tau) = \frac{1}{L^2}\left[-\frac{33}{4}\frac{\tilde{E}^2}{a^8} + \frac{3}{2}\frac{E}{a^4}\right]. \tag{5.59}$$

The E dependence of these equations slightly complicates the analysis of these equations, and hence we begin with the simplest case in which $E = 0$.

Case 1 $E = 0$:

When $E = 0$ (the static limit in which the probe has zero kinetic energy at infinity (see Fig. 5.1)) only the term proportional to a^{-8} survives in (5.58) and (5.59). Recall that when r_H vanishes the potential V_{eff}^{τ} is flat. Furthermore, since $\tilde{E} \propto r_H^4 = 0$, it follows from (5.59) that $M^2(\tau) = 0$ in this limit: as expected, a BPS probe brane with zero energy in AdS$_5$ has no dynamics and is completely stable.

When $r_H \neq 0$, $M^2(\tau) < 0 \; \forall \tau$, and the solution of (5.58) is

$$a(\tau)^4 = a_i^4 \pm \frac{2a_H^4}{L}(\tau - \tau_i). \tag{5.60}$$

Here $a_i \geq a_H \equiv r_H/L$ is the initial position of the brane at $\tau = \tau_i$, and the choice of sign determines whether the brane is moving radially inward ($-$) or outward ($+$): this is a question of initial conditions. Let $R_h = 1/|Ha|$ denote the (comoving) Hubble radius. Then it follows from (5.59) and the definition of k_c^2 in (5.57) that

$$\frac{1}{\lambda_c} \sim |k_c(\tau)| \sim |Ha| = \frac{1}{R_h}, \tag{5.61}$$

where we neglect numerical factors of order 1. Thus the critical wavelength is $\lambda_c \sim R_h$. (Notice that R_h is minimal at a_H and increases with a.)

For *superhorizon* modes $\lambda \gg R_h$ or $|k| \ll |k_c|$, and in this limit the perturbation equation (5.56) becomes

$$\varphi_{k,\tau\tau} - \frac{k_c^2(\tau)}{a^2}\varphi_k = 0. \tag{5.62}$$

On inserting solution (5.60) into k_c^2 one obtains

$$\phi_k = \frac{\varphi_k}{a^{3/2}} = A_k a^4 + B_k a^{-3}, \tag{5.63}$$

(where the constants A_k and B_k are determined by the initial conditions). Hence if the brane moves radially outward the superhorizon modes grow as $a^4 \propto \tau$. If the brane is contracting they grow as a^{-3}. In the near extremal limit, $r_H \ll L$ or $a_H \ll 1$, the amplitude of these superhorizon modes can become very large, suggesting that they are unstable. Of course our linear analysis will break down when ϕ becomes too large.

Consider now *subhorizon* modes $\lambda \ll R_h$ or $|k| \gg |k_c|$. Then (5.56) is just $\varphi_{k,\tau\tau} + \left(k^2/a^2\right)\varphi_k = 0$. However, in this case it is much easier to solve the equation in conformal time η where the factor of a^{-2} is no longer present. We anticipate the result from paragraph 5.3.3: it is

$$\phi_k = A_k\frac{\mathrm{e}^{ik\eta}}{a} + B_k\frac{\mathrm{e}^{-ik\eta}}{a}. \tag{5.64}$$

For an outgoing brane a increases and subhorizon modes are stable. For an ingoing brane a decreases, and the amplitude of the perturbation becomes very large in the near extremal limit. (Note that, as the brane expands, superhorizon modes eventually become subhorizon, and similarly, on a contracting brane, subhorizon modes become superhorizon.)

To conclude, when $r_H \neq 0$, $E = 0$, and the brane expands, superhorizon modes are unstable while subhorizon modes are stable. For a contracting brane, and in the near extremal limit, both super and subhorizon modes are unstable.

Case 2 $E \neq 0$:

When the energy of the brane is non zero the situation is more complicated. Notice first from (5.59) that $M^2(\tau)$ has one zero at $a = a_c$ given by

$$a_c^4 = \frac{11\tilde{E}^2}{2E}. \tag{5.65}$$

Hence $M^2(\tau)$ is negative when $a < a_c$ and positive for $a > a_c$ (see Fig. 5.5). However, since a_c is E dependent, there may be ranges of E for which the negative mass region is hidden within the black-hole horizon. Indeed, we find

$$a_c \leq a_H \qquad \Longleftrightarrow \qquad E_- \leq E \leq E_+, \tag{5.66}$$

where

$$E_\pm = \frac{a_H^4}{22}(13 \pm 4\sqrt{3}). \tag{5.67}$$

The situation is shown schematically in Fig. 5.5.

Figure 5.5: The curve represents a_c, the zero of $M^2(\tau)$, as a function of the energy E as given in Eq. 5.65. Below the curve the effective mass squared is negative, above it is positive. For $E < E_-$ and $E > E_+$ the $M^2(\tau)$ already becomes negative outside the horizon, whereas for energies within the interval E_-, E_+ the $M^2(\tau) < 0$ region is hidden within the horizon. The parameters chosen are $q = 1$, $r_H = 1$, and $L = 1$.

Now consider H^2 given in Eq. (5.58). The two terms are of equal magnitude when $a = \tilde{a}_c = (\tilde{E}^2/2E)^{1/4} \sim a_c$. Thus when $a \ll a_c$ (and hence in the regions in which $M^2 < 0$ in Fig. 5.5), the dominant term in H^2 is the one proportional to a^{-8}. The system is therefore analogous to the one considered above when $E = 0$, and for superhorizon modes the solution is given in (5.63): for an outgoing brane $\phi_k \sim a^4$. When $E \stackrel{>}{\sim} E_+$ or $E \stackrel{<}{\sim} E_-$, these regimes extend down to the black-hole horizon: thus in the near extremal limit the contracting brane will again be unstable since $\phi_k \sim a^{-3}$.

When $a \gg a_c$ (and hence in the regimes in which $M^2 > 0$ in Fig. 5.5), the dominant term in H^2 is $\propto a^{-4}$ so that

$$a(\tau)^2 = a_i^2 \pm 2\frac{\sqrt{2E}}{L}(\tau - \tau_i) \tag{5.68}$$

and

$$k_c^2(\tau) = -M^2(\tau)a^2 = -\frac{3}{2}\frac{E}{L^2a^2}. \tag{5.69}$$

On *superhorizon* scales the mode equation is

$$\varphi_{k,\tau\tau} + \frac{3}{2}\frac{E}{L^2a^4}\varphi_k = 0. \tag{5.70}$$

At first sight one might expect the solution to this equation to be stable since $M^2 > 0$. However, surprisingly, it is not. (Indeed, below we will see that in conformal time the effective mass is actually negative in this regime.) A change of variables to $u = a^2$ shows that the solution of (5.70) is

$$\varphi_k = A_k a^{3/2} + B_k a^{1/2} \tag{5.71}$$

which grows as $\tau^{3/4}, \tau^{1/4}$ respectively. Finally,

$$\phi_k = A_k + B_k a^{-1}. \tag{5.72}$$

For E within the band $E_- \lesssim E \lesssim E_+$, the solution (5.72) for the modes is valid for all a so that *superhorizon* modes grow as a^{-1} as the brane approaches the black-hole horizon.

When $E \gtrsim E_+$ or $E \lesssim E_-$ these solutions are valid for $a \gg a_c$. Thus for an expanding brane ϕ_k tends to a constant value. For a contracting brane, the term $\propto a^{-1}$ could become important, although for small enough a the relevant regime is that considered above in which case the solution is given by (5.63) and the superhorizon modes grow as a^{-3}.

For *subhorizon* modes, the solution is still as given in (5.64).

BPS anti-branes: $q = -1$

Now the Friedmann equation (5.29) and effective mass $M^2(\tau)$ become

$$H^2 = \frac{1}{L^2}\left[\frac{\tilde{E}^2}{a^8} - \frac{2E}{a^4}\right], \tag{5.73}$$

$$M^2(\tau) = -\frac{1}{L^2}\left[\frac{33}{4}\frac{\tilde{E}^2}{a^8} + \frac{3}{2}\frac{E}{a^4}\right] \tag{5.74}$$

so that M^2 is always negative, independently of E. Note that $H^2 > 0$ for $a < \tilde{a}_c$ where $\tilde{a}_c = (\tilde{E}^2/2E)^{1/4}$. However, since $\tilde{E} = E + a_H^4/2$ for anti-branes, it follows that $\tilde{a}_c \geq a_H$ for *all* E (*i.e.* there are no energy bands to consider in the case of anti-branes). When $a \ll \tilde{a}_c$, $H^2 \propto M^2 \propto a^{-8}$ and once again this is analogous to the case studied above for $E = 0$: *superhorizon* modes grow as a^4, and in the near extremal limit the *subhorizon* modes on an ingoing brane are unstable.

Non BPS branes: $q \neq \pm 1$

Here we shall only briefly discuss the case $q^2 > 1$ for large a. Now, independently of E, there is a cosmological constant dominated regime (see Eq. (5.29)). There the solution for the scale factor is

$$a(\tau) = a(\tau_i)\mathrm{e}^{\pm\sqrt{\Lambda_4/3}(\tau-\tau_i)} \qquad \text{where} \qquad \frac{\Lambda_4}{3} \equiv \frac{q^2-1}{L^2}. \tag{5.75}$$

In this regime, however, M^2 is negative with

$$M^2(\tau) = -\frac{25}{12}\Lambda_4 \tag{5.76}$$

and $R_h = 1/|Ha| = (\Lambda_4/3)^{-1/2}a^{-1}$.

For *subhorizon* modes ($\lambda \ll R_h$) the solution for φ_k is again given by (5.64). For *superhorizon* modes, and considering an outgoing brane, there is an exponentially growing unstable mode

$$\phi_k = A_k \mathrm{e}^{\sqrt{\Lambda_4/3}(\tau-\tau_i)} = A_k a. \tag{5.77}$$

Hence, this non BPS brane is unstable for large a. It is not clear to us why the acceleration due to the positive cosmological constant does not rather stretch the perturbations away.

5.3.3 Comments on an analysis in conformal time η

It is instructive to carry out a similar analysis in conformal time η rather than cosmic time τ, and we comment briefly on it here. In conformal time and transformed to Fourier space, Eq. (5.38) becomes

$$\phi_{k,\eta\eta} + 2\mathcal{H}\phi_{k,\eta} + (k^2 + a^2m^2)\phi_k = 0, \tag{5.78}$$

where $\mathcal{H} = aH$. The friction term can be eliminated by a change of variables to $\psi = a\phi$, and the above equation becomes

$$\psi_{k,\eta\eta} + \left(k^2 - k_c^2(\eta)\right)\psi_k = 0, \tag{5.79}$$

where

$$k_c^2(\eta) = -\mathcal{M}^2(\eta)$$

and

$$
\begin{aligned}
\mathcal{M}^2(\eta) &= a^2 m^2 - a_{\eta\eta}/a \\
&= gm^2 + \frac{1}{2}\left[-\frac{g''}{g}R_\eta^2 + \frac{1}{2}\left(\frac{g'}{g}\right)^2 R_\eta^2 - \frac{g'}{g}R_{\eta\eta}\right] \\
&= -\frac{(E-C)^2}{fg^2h}\left[\frac{1}{2}\frac{f'}{f}\frac{g'}{g} - \frac{g''}{g} + 3\left(\frac{g'}{g}\right)^2 + \frac{1}{2}\frac{g'}{g}\frac{h'}{h} + \frac{5}{2}\frac{g'}{g}\frac{C'}{E-C}\right. \\
&\left. + \left(\frac{C'}{E-C}\right)^2\right] \\
&+ \frac{g}{2h}\left[\frac{f''}{f} - \frac{1}{2}\left(\frac{f'}{f}\right)^2 + \frac{3}{2}\frac{f'}{f}\frac{g'}{g} - \frac{1}{2}\frac{f'}{f}\frac{h'}{h} + \frac{g''}{g} - \frac{1}{2}\frac{g'}{g}\frac{h'}{h}\right].
\end{aligned} \tag{5.80}
$$

Specializing to AdS$_5$-S yields

$$\mathcal{M}^2(\eta) = -\frac{1}{L^2}\left[\frac{10\tilde{E}^2}{a^6} + 6(q^2-1)a^2\right]. \tag{5.81}$$

Notice that in conformal time and for $|q| \geq 1$, $\mathcal{M}^2(\eta)$ is *always* negative independently of E. From this, one can immediately see the instability for small k in Eq. (5.71), even though $M^2(\tau)$ can be positive in that case. It is clear that the results on brane (in)stability must be independent of whether or not the analysis is carried out in η or τ time. We will see that this is indeed the case: the reason is that not only the sign of the effective mass squared but also its functional dependence on time determine the stability properties. We now summarize briefly some of the aspects that differ between the τ and η analysis.

Consider the simplest case: $q = +1$ and $E = 0$. The solution of the (conformal time) Friedmann equation is $a^3 = a_i^3 \pm 3a_H^2(\eta - \eta_i)/2L$, and $k_c(\eta) \sim |\mathcal{H}| = 1/R_h$. For *superhorizon* modes $|k| \ll |k_c|$ Eq. (5.79) reduces to $\psi_{k,\eta\eta} - k_c^2(\eta)\psi_k = 0$. Given $a(\eta)$ and hence $k_c(a(\eta))$ it is straightforward to find the solution which is, as expected, exactly that given in (5.63). For *subhorizon* modes, $|k| \gg |k_c|$, the solution was given in (5.64).

Consider now $q = +1, E > 0$. Recall that in the τ time analysis both $M^2(\tau)$ and H^2 contained terms in a^{-4} and a^{-8} and, in particular, there was a regime in which $M^2(\tau)$ was positive and proportional to $a^{-4} \propto H^2$. In η time, however, $\mathcal{H} \propto a^{-6} + a^{-2}$ with $\mathcal{M}^2$ is always negative and $\propto -a^{-6}$. Thus while the $a \ll a_c$ regime reduces to that discussed above for $E = 0$, the $a \gg a_c$ regime is a little less clear. There $\mathcal{H}^2 \sim a(\eta)^{-2}$, but $\mathcal{M}^2 \sim -a(\eta)^{-6}$. Thus

$$a(\eta) = a_i \pm \frac{\sqrt{2E}}{L}(\eta - \eta_i) \tag{5.82}$$

and

$$k_c(\eta)^2 = -\mathcal{M}(\eta)^2 = \frac{10\tilde{E}^2}{L^2 a^6}. \tag{5.83}$$

Now $|k_c(\eta)| \sim |\mathcal{H}|^3 L^2 = L^2/R_h^3$, and so one can no longer identify the critical wavelength with the Hubble radius. For $|k| \ll |k_c|$ Eq. (5.79) reduces to $\mathrm{d}^2\psi_k/\mathrm{d}a^2 - (5\tilde{E}^2/E)(\psi_k/a^6) = 0$. The solution is expressed in terms of Bessel functions which, however, show exactly the same behavior as (5.72): namely $\phi_k = \psi_k/a$ tends to a constant as $a \to \infty$. The other limit $a \to 0$ is not relevant as the above equation is only valid for $a \gg a_c$.

We do not discuss further the case of $q = -1$ and $q \neq 1$ since the results obtained in this approach are exactly as discussed in paragraphs 5.3.2 and 5.3.2.

5.4 Bardeen potentials

So far we have discussed the evolution of ϕ which is the magnitude of the brane perturbation as seen by a 5-dimensional observer comoving with the brane. For an observer living *on* the brane, the perturbed brane embedding gives rise to perturbations about the FRW geometry. Recall (see Eq. (5.9)) that for the unperturbed brane

$$\begin{aligned} \mathrm{d}s_4^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -(f(R) - h(R)\dot{R}^2)\mathrm{d}t^2 + g(R)\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} \\ &\equiv -n^2(t)\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}. \end{aligned} \tag{5.84}$$

The scale factors $n^2(t)$ and $a^2(t)$ pick up their time dependence through $R(t)$, for instance $a^2(t) = g(R(t))$. In this section we calculate $\delta g_{\mu\nu}$ resulting from the perturbed embedding (5.32) and relate it to the Bardeen potentials.

Initially, rather than using the covariant form (5.32), let us write more generally

$$\tilde{X}^0(t,\vec{x}) = t + \zeta^0(t,\vec{x}), \tag{5.85}$$

$$\tilde{X}^i(t,\vec{x}) = x^i + \zeta^i(t,\vec{x}), \tag{5.86}$$

$$\tilde{X}^4(t,\vec{x}) = R(t) + \epsilon(t,\vec{x}). \tag{5.87}$$

Below we will see that the perturbations ζ^i do not enter into the two scalar Bardeen potentials which correspond to the two degrees of freedom ζ^0 and ϵ. This is expected since perturbations parallel to the brane are not physical and can be removed by a coordinate transformation [47]. Then only right at the end will we set $\zeta^0/n^0 = \epsilon/n^5 = \phi$. We will find that the two Bardeen potentials are proportional to each other and to ϕ.

By definition, the perturbed induced metric is given by

$$\begin{aligned} \tilde{g}_{\mu\nu} &= g_{\mu\nu} + \delta g_{\mu\nu} \\ &= G_{MN}(X+\delta X)\frac{\partial}{\partial x_\mu}(X^M + \delta X^M)\frac{\partial}{\partial x_\nu}(X^N + \delta X^N). \end{aligned} \tag{5.88}$$

Evaluating $\delta g_{\mu\nu}$ to first order for the perturbed embedding (5.85)-(5.87) and the general bulk metric (5.5), one obtains

$$\begin{aligned}
\delta g_{00} &= \epsilon(-f' + h'\dot{R}^2) + 2(-\dot{\zeta}^0 f + \dot{\epsilon} h \dot{R}), & (5.89)\\
\delta g_{0i} &= -(\partial_i \zeta^0) f + \dot{\zeta}^i g + (\partial_i \epsilon) h \dot{R}, & (5.90)\\
\delta g_{ij} &= \epsilon g' \delta_{ij} + (\partial_i \zeta_j + \partial_j \zeta_i) g. & (5.91)
\end{aligned}$$

Note the terms proportional to ϵ come from the Taylor expansion of $G_{MN}(X+\delta X)$ in (5.88) to first order.

In the usual way, the perturbed line element on the brane is written as

$$\mathrm{d}\tilde{s}_4^2 = -n^2(1+2A)\mathrm{d}t^2 - 2anB_i \mathrm{d}t\mathrm{d}x^i + a^2(\delta_{ij} + h_{ij})\mathrm{d}x^i \mathrm{d}x^j, \tag{5.92}$$

where $n(t)$ and $a(t)$ are defined in (5.84), and as usual vectors are decomposed into a scalar part and a divergenceless vector component *e.g.*

$$B_i = \partial_i B + \bar{B}_i \tag{5.93}$$

with $\partial^i \bar{B}_i = 0$. We will use a similar decomposition for ζ^i defined in (5.86) as well as the usual one for tensor perturbations. Thus from (5.89)-(5.91) we have

$$\begin{aligned}
A &= \frac{1}{n^2}\left[\frac{\epsilon}{2}(f' - h'\dot{R}^2) + (\dot{\zeta}^0 f - \dot{\epsilon} h \dot{R})\right],\\
B &= \frac{1}{an}\left[\zeta^0 f - \dot{\zeta} a^2 - \epsilon h \dot{R}\right],\\
\bar{B}_i &= -\frac{a}{n}\dot{\bar{\zeta}}_i,\\
C &= \frac{\epsilon}{2}\left(\frac{g'}{g}\right),\\
E &= \zeta,\\
\bar{E}_i &= \bar{\zeta}_i,\\
\bar{\bar{E}}_{ij} &= 0,
\end{aligned} \tag{5.94}$$

where we have used standard notation defined *e.g.* in [129]. By considering coordinate transformations on the brane and doing standard 4-dimensional perturbation theory one can define the usual two Bardeen potentials, as well as the brane vector and tensor metric perturbations. For the first Bardeen potential we find, after some algebra,

$$\begin{aligned}
\Phi &= -C + \frac{\dot{a}}{n}\left(B + \frac{a}{n}\dot{E}\right)\\
&= \left(\frac{\dot{a}}{a}\right)\frac{f}{n^2}\frac{1}{\dot{R}}\left[\zeta^0 \dot{R} - \epsilon\right].
\end{aligned} \tag{5.95}$$

Notice that all terms containing ζ^i in B and E have cancelled as expected since they are not physical degrees of freedom. Similarly,

$$\begin{aligned}\Psi &= A - \frac{1}{n}\partial_t\left(aB + \frac{a^2}{n}\dot{E}\right)\\ &= \frac{1}{n^2}\frac{1}{\dot{R}}\left[\zeta^0\dot{R} - \epsilon\right]\left[f'\dot{R} - f\left(\frac{\dot{n}}{n}\right)\right].\end{aligned} \tag{5.96}$$

The important point to notice in this second case is not only the absence of ζ^i, but that all *derivatives* of the perturbations ζ^0 and ϵ (which appear in A) have also cancelled. Hence we will find that the Bardeen potentials are proportional to ϕ only and not to any of its derivatives. Finally, the gauge invariant vector and tensor perturbations are identically zero.

We now set

$$\epsilon = n^4\phi, \qquad \zeta^0 = n^0\phi, \tag{5.97}$$

(where n^4 is the radial component of the unit normal n^M to the brane) in order to make contact with the covariant formalism of section 5.3. Then the combination that appears in both Ψ and Φ is

$$\zeta^0\dot{R} - \epsilon = -\left(\frac{n^2(t)}{f}n^4\right)\phi, \tag{5.98}$$

where n^4 is the radial component of the normal to the unperturbed brane. Thus

$$\Phi = -\left(\frac{\dot{a}}{a}\right)\frac{n^4}{\dot{R}}\phi\,, \qquad \Psi = \left(\frac{f'}{f}\dot{R} - \frac{\dot{n}}{n}\right)\frac{n^4}{\dot{R}}\phi \tag{5.99}$$

which, on going to AdS$_5$-S and using the expression for $\dot{R}^2$ in (5.14), yields

$$\Phi = -\frac{(E - C(a))}{a^4}\left(\frac{\phi}{L}\right) = -\left(\frac{\tilde{E}}{a^4} + q\right)\left(\frac{\phi}{L}\right), \tag{5.100}$$

$$\Psi = 3\Phi + 4q\left(\frac{\phi}{L}\right). \tag{5.101}$$

Even though there are no anisotropic stresses, the Bardeen potentials here are not equal. This is because of the absence of Einstein's equations in mirage cosmology[6]. We see that for superhorizon modes on an expanding brane (for which, from section 5.3, $\phi_k \propto a^4$), we also have $\Phi_k \propto a^4$. Similarly, Φ_k also grows rapidly for a brane falling into the black-hole horizon.

To obtain a true (*i.e.* gauge invariant) measure of the deviation from the FRW metric, it is useful to look at the ratio of the components of the perturbed Weyl tensor and the background Riemann tensor which in the FRW case is roughly given by $(k\eta)^2|\Phi_k + \Psi_k|$, see [52]. For $\Phi_k \propto a^4$ this ratio grows, because $a \sim \eta^{1/3}$ when $\mathcal{H}^2 \sim a^{-6}$.

[6]In standard 4-dimensional cosmological perturbation theory, Einstein's equations impose that the (scalar) anisotropic stress is proportional to $\Psi - \Phi$.

5.5 Conclusions

In this paper we have studied the evolution of perturbations on a moving D3-brane coupled to a bulk 4-form field, focusing mainly on an AdS_5-Schwarzschild bulk. For an observer on the unperturbed brane, this motion leads to FRW expansion or contraction with a scale factor $a \propto r$. We assumed that there is no matter on the brane and ignored the back-reaction of the brane onto the bulk. Instead, we aimed to investigate the growth of perturbations due only to motion, and also to study the stability of moving D3-branes. For such a probe brane, the only possible perturbations are those of the brane embedding. The fluctuations about the straight brane world sheet are described by a scalar field ϕ which is the proper amplitude of a 'wiggle' seen by an observer comoving with the unperturbed brane. Following the work of [64, 72, 34] we derived an equation of motion for ϕ, and investigated whether small fluctuations are stretched away by the expansion, or, on the other hand, whether they grow on a contracting brane. The equation for ϕ is characterized by an effective mass squared and we noted that if this mass is positive, the system is not necessarily stable: indeed, in section 5.3 we discussed a regime in which the effective mass squared is positive in brane time but negative in conformal time, and therefore the perturbations grow. Another important factor in the evolution of ϕ is the time dependence of that mass.

In section 5.3 we found that on an expanding BPS brane with total energy $E = 0$, superhorizon modes grow as a^4, whereas subhorizon modes decay and hence are stable. For a contracting brane, on the contrary, both super and subhorizon modes grow as a^{-3} and a^{-1} respectively. These fluctuations become large in the near extremal limit, $a_H \ll 1$. We therefore concluded that the brane becomes unstable (*i.e.* the wiggles grow) as it falls into the black-hole. We also discussed the case $E > 0$ for BPS branes and BPS anti-branes. Non BPS branes were found to be unstable at late times when a positive cosmological constant dominates.

We have discussed the evolution of the fluctuations ϕ as measured by a five-dimensional observer moving with the unperturbed brane. However, for an observer at rest in the bulk, the magnitude of the perturbation is given by a Lorentz contraction factor times the proper perturbation ϕ. (For a flat bulk space-time this was pointed out in [64].) Hence, if perturbations grow for the 'comoving' observer, they do not necessarily grow for an observer at rest in the bulk.

Finally, the fluctuations around the unperturbed world-sheet generate perturbations in the FRW universe. In section 5.4 we discussed these perturbations from the point of view of a 4-dimensional observer living on the perturbed brane. We calculated the Bardeen potentials Φ and Ψ which were both found to be proportional to ϕ. Furthermore, we saw that the ratio 'Weyl to Riemann' which, expressed in terms of Φ and Ψ, gives a gauge invariant measure for the 'deviation' from the FRW metric, also grows.

A limitation of this work is that the back-reaction of the brane onto the bulk was neglected. One may wonder whether inclusion of the back-reaction could stabilize ϕ. To answer that question, recall that the setup we have analyzed here corresponds, in the junction condition approach, to one in which Z_2 symmetry across the brane is broken. Then the brane is at the interface of two AdS$_5$-S space-times, and its total energy is related to the difference of the respective black-hole masses: $\tilde{E} \propto M_+ - M_-$. Perturbation theory in such a non Z_2-symmetric self-interacting case has been set up in [129], though it is technically quite complicated. However, in the future we hope to try to use that formalism to include the back-reaction of the brane onto the bulk.

It would be interesting to extend this analysis to branes with n codimensions: in this case one has to consider n scalar fields – one for each normal to the brane. Formalisms to treat this problem have been developed in [16, 73]. In that case the equations of motion for the scalar fields are coupled, and it becomes a complicated task to diagonalize the system.

Finally, it would also be interesting to consider non zero $F_{\mu\nu}$, and hence the effect of perturbations in the radiation on the brane.

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Chapter 6

Cosmology on a back-reacting brane

6.1 Introduction

In the previous chapters 4 and 5, we identified our universe with a probe 3-brane moving in a background geometry without including the back-reaction. However, depending on the coupling constant of the theory, a brane can be 'heavy', in which case the assumption that the back-reaction can be neglected is not valid anymore[1]. To take the back-reaction into account, one has to resort to Einstein's equations. There are two possible ways to do that, which are however limited to the case that the co-dimension is one. In the following, we therefore consider a 3-brane in a 5-dimensional bulk, and assume that the bulk dynamics are governed by the 5-dimensional Einstein equations. The first method is to use the junction conditions (3.50), such that the dynamics on the brane includes effects from the back-reaction. The second method is to calculate the induced 4-dimensional Einstein equations by the relations of Gauss, Codazzi, and Mainardi (3.35) and (3.38). We present the two approaches in Sec. 6.3 and 6.4, respectively.

In mirage cosmology, we have used a static background motivated by supergravity, and the dynamics were generated by letting the brane move. Here, we shall use a set-up in which the 3-brane is at rest at the price of having a time-dependent bulk. It has been shown by the authors of Ref. [142], that the two viewpoints are equivalent, in that the AdS_5 metric is related to the time-dependent metric by a coordinate transformation. This reconciles the supergravity compactification $\mathrm{AdS}_5 \times \mathrm{S}^5$ with the Hořawa-Witten compactification.

We start this chapter by giving a short introduction to the model of Randall and Sundrum. In Sec. 2.7 we presented an idea to solve the hierarchy problem with 'large' extra-dimensions, in which the 4-dimensional Planck mass is related to the fundamental scale of gravity by (see Eq. (2.74))

$$M_4^2 = M_D^{2+n} V_n, \tag{6.1}$$

where V_n is the volume of some compact space, for instance an n-sphere S^n. For this idea to work, it is crucial that the higher-dimensional space-time has a product structure $\mathcal{M}^D = \mathcal{M}^4 \times S^n$, *i.e.* that the metric is factorizable. Otherwise, it would be impossible to carry out the dimensional reduction (2.72). The model of large extra-dimensions has been criticized by Randall and Sundrum, for it introduces a new hierarchy between the electroweak scale M_{EW} (TeV) and the compactification scale M_c, which is the energy associated with the first excited Kaluza-Klein state. While it considerably relieves the hierarchy problem, the Randall-Sundrum model suffers from a fine-tuning problem which, in a cosmological context, is very severe, as we point out in the article 'Dynamical instabilities of the Randall-Sundrum model' in Chap. 7. Nevertheless, the Randall-Sundrum model is a prototype in brane cosmology, and so it is worth mentioning it.

[1] Explicit mass formulae for p-branes are given in Eqs. (11.18)-(11.22) and Eqs. (11.23)-(11.27).

6.2 The Randall-Sundrum model

The Randall-Sundrum (RS) model is inspired from Hořava-Witten compactification (paragraph 2.5.3), in which space-time is 5-dimensional and two 3-branes are located at orbifold fixed points. One of the branes is our 'visible' universe and supports the standard model matter fields, the other brane will be referred to as 'hidden' brane. Mass scales on the two branes are related to each other by an exponential factor arising from the particular form of the background metric. Thus, by an exponential suppression, small scales on one brane can be generated from large scales on the other. In particular, this mechanism offers a solution to the hierarchy problem in which all scales are derived from a single scale.

6.2.1 Warped geometry solution

In next two paragraphs, we briefly discuss the first model of Randall and Sundrum, Ref. [128]. Let us consider a 5-dimensional space-time $\mathcal{M}^4 \times S^1/Z_2$, parameterized by coordinates (x^μ, ϕ), where ϕ lies in the interval $[-\pi, \pi]$ with the identification $\phi \leftrightarrow -\phi$. Two 3-branes parallel to $\mathcal{M}^4$ reside at the orbifold fixed points $\phi = 0, \pi$. They are the boundaries of the RS space-time, which is the half space $\{0 \leq \phi \leq \pi\}$.

We choose an embedding, such that the induced metrics on the branes are, respectively

$$g^{\text{hid}}_{\mu\nu}(x^\mu) = G_{\mu\nu}(x^\mu, \phi = 0), \quad g^{\text{vis}}_{\mu\nu}(x^\mu) = G_{\mu\nu}(x^\mu, \phi = \pi), \tag{6.2}$$

where $G_{\mu\nu}$ denote the components of the 5-dimensional metric along $\mathcal{M}^4$. The action of the RS model is[2,3]

$$\begin{aligned} S_5 &= S_{\text{gravity}} + S_{\text{vis}} + S_{\text{hid}}, \\ S_{\text{gravity}} &= \frac{M_5^3}{2} \int \mathrm{d}^4x \int_{-\pi}^{\pi} \mathrm{d}\phi \, r_c \sqrt{-G}(R - \Lambda), \\ S_{\text{vis}} &= \int \mathrm{d}^4x \, \sqrt{-g_{\text{vis}}}(\mathcal{L}_{\text{vis}} - V_{\text{vis}}), \\ S_{\text{hid}} &= \int \mathrm{d}^4x \, \sqrt{-g_{\text{hid}}}(\mathcal{L}_{\text{hid}} - V_{\text{hid}}). \end{aligned} \tag{6.3}$$

Here, Λ is a 5-dimensional cosmological constant, $\mathcal{L}_{\text{vis}}$ is the Lagrangian density of the standard model fields, and V_{vis} is the tension of the brane. The corresponding Einstein equations can be found from Eqs. (7.49)-(7.52) in the article in Chap. 7 by setting all time-dependent terms to zero. Here, we are looking for a static solution respecting 4-dimensional Poincaré invariance,

$$\mathrm{d}s_5^2 = \mathrm{e}^{2R(\phi)} \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + r_c^2 \mathrm{d}\phi^2, \tag{6.4}$$

[2]For consistency with the notation in this thesis as well as with that of most other authors, we have normalized R and Λ differently than in the original RS paper: $\Lambda_{RS} = (M_5^3/2) r_c \Lambda$, $M_{RS}^3 = (M_5^3/4) r_c$.

[3]In the article in Chap. 7, we work in units where $M_5^3/2 \equiv 1/2\kappa_5^2 = 1$.

where r_c denotes the compactification radius. In contrast to the large extra-dimensions scenario, this metric is not factorizable: there is a so-called 'warp factor' multiplying the Minkowski metric along the brane. This is the crucial idea of the RS model. Inserting the ansatz (6.4) into the equations of motion, one finds[4]

$$R(\phi) = -r_c|\phi|\sqrt{\frac{-\Lambda}{12}}. \tag{6.5}$$

For later purpose we define a parameter $\alpha \equiv -\sqrt{\frac{-\Lambda}{12}}$ which represents a mass scale set by the cosmological constant in the bulk. This will become important in the next paragraph. From Eq. (6.5), clearly Λ must be negative, which reflects the fact that the ansatz (6.4) together with Eq. (6.6) is a parametrization of AdS_5 (see also (2.41)). There is a second equation of motion to satisfy, containing Dirac delta functions from the boundaries. This requires a fine-tuning between the brane tension and the cosmological constant in the bulk,

$$V_{\mathrm{hid}} = -V_{\mathrm{vis}} = \frac{M_5^3}{2}\sqrt{-12\Lambda}. \tag{6.6}$$

The first equality arises from the requirement that the sum of the tensions on an orbifold must be zero. It turns out, that the tension of the brane we are living on is negative, which is rather unphysical. This will be changed in the second model of RS.

Notice again, that the RS model is purely static. In Chap. 7, we construct a cosmological, *i.e.* dynamical version of it and perform a 5-dimensional perturbation analysis. We find, that the model is unstable, as soon as the fine-tuning relation (6.6) is perturbed. Intuitively, this is clear, as a 'de-tuning' corresponds to turning on a 4-dimensional cosmological constant on the brane (see also Sec 6.4).

6.2.2 Scales and the hierarchy problem

Let us write the warped geometry solution as

$$\mathrm{d}s_5^2 = G_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + r_c^2\mathrm{d}\phi^2 = \mathrm{e}^{-2r_c|\phi||\alpha|}\left(\eta_{\mu\nu} + h_{\mu\nu}\right)\mathrm{d}x^\mu \mathrm{d}x^\nu + r_c^2\mathrm{d}\phi^2, \tag{6.7}$$

where $h_{\mu\nu}(x^\mu)$ parameterizes small fluctuations around the Minkowski background and corresponds to the massless zero mode in the Kaluza-Klein expansion[5]. By comparison with Eq. (6.2), one finds

$$g_{\mu\nu}^{\mathrm{hid}} = \eta_{\mu\nu} + h_{\mu\nu}, \quad g_{\mu\nu}^{\mathrm{vis}} = \mathrm{e}^{-2r_c\pi|\alpha|}g_{\mu\nu}^{\mathrm{hid}}. \tag{6.8}$$

Therefore, any two mass scales on the branes are inversely related by

$$M^{\mathrm{vis}} = \mathrm{e}^{+r_c\pi|\alpha|}M^{\mathrm{hid}}. \tag{6.9}$$

[4] See also Eq. (7.27) with the identification $y = r_c\phi$.

[5] We are not considering perturbations in r_c, but simply assume that the radius of the extra-dimensions is fixed to the vacuum expectation value of some modulus field.

This is readily understood, for instance, by transforming the action of a canonically normalized scalar field from the visible to the hidden brane or vice versa. Eq. (6.9) is the key to the solution of the hierarchy problem: assume that the 5-dimensional Planck mass, M_5, is a mass parameter on the hidden brane, giving the 'true strength' of gravity. Set $M_5 \simeq M_{EW} \simeq 1\,\mathrm{TeV}$, to put gravity and gauge interactions on an equal footing. If one now requires that the corresponding mass parameter on the visible brane is $10^{18}\,\mathrm{GeV} = 10^{15}\,\mathrm{TeV}$, one obtains

$$10^{15}\,\mathrm{TeV} = \mathrm{e}^{+r_c \pi |\alpha|}\,\mathrm{TeV}. \tag{6.10}$$

In order not to create just another hierarchy, we have to set the mass parameter α also to the TeV scale. Consequently, $r_c \simeq 10/\mathrm{TeV}$, such that the mass scale associated with the compactification radius, $1/r_c$ is also roughly at the TeV scale. Thus, in this scenario, there is no large hierarchy between the scales: $M_5 \simeq M_{EW} \simeq |\alpha| \simeq 1/r_c \simeq 1\,\mathrm{TeV}$, and the huge value of the Planck mass in our visible universe is explained by an exponential factor. Conversely, one can view $M_4 \simeq 10^{18}\,\mathrm{GeV}$ as the fundamental scale, and the TeV as a derived scale, which is done so in the original formulation of RS. Regarding the solution of the hierarchy problem, the RS proposal is clearly an improvement compared to the scenario of large extra-dimensions. There, the relation between the 'effective' Planck mass and the 'fundamental' scale is $M_4^2 = M_5^3 r_c^n$ (with $n = 1$ here). Using the same values for M_4 and M_5 as above, one obtains $1/r_c = 10^{-30/n}\,\mathrm{TeV}$, *i.e.* a huge hierarchy between the compactification scale and the TeV.

As an experimental signature of the RS model, one expects new excitations with TeV energies, since $1/r_c \simeq 1\,\mathrm{TeV}$ is the energy of the first Kaluza-Klein state. Such energies should be accessible to future collider experiments.

6.2.3 Non compact extra-dimension

Due to the warp factor, distances parallel to the brane become rapidly very small as one moves away from the location of the brane into the fifth dimension. Effectively, this dimension becomes 'invisible' for an observer on the brane. This hints at the possibility that the extra-dimension may even be infinite, still leading to an acceptable 4-dimensional phenomenology. Subsequently, RS proposed a second model [127], which emerges from the first one by taking the limit $r_c \to \infty$. In the orbifold set up, this is accomplished by pushing one brane to infinity, thereby decompactifying the S^1. The remaining brane, with positive tension, is now identified with our universe. For this second model, the key observation is, that the curved background (6.7) can support a single normalizable bound state of the higher-dimensional graviton. Since this bound state is localized at the position of the brane, it plays the role of the 4-dimensional graviton, and the usual $1/r^2$ Newton law is recoverd. In addition, there is a continuum of KK states, because the extra-dimension is infinite. Even though there is no gap between the continuum and the ground state, 4-dimensional gravity is very well approximated, because the continuum states contribute only weakly. In fact, in Ref. [127], it is shown that n non compact extra-dimensions are compatible with the $1/r^2$ form of Newton's law. This takes away some observational pressure on KK type models, discussed in Sec. 2.7. Unfortunately, it seems difficult to embed the phenomenological approach of RS in string theory.

Because the RS solution is static, it cannot be a viable model for brane cosmology. The next step is therefore to look for time-dependent solutions of the 5-dimensional Einstein's equations.

6.3 Brane cosmological equations

We shall start from an action similar to the one of the RS model. This time, we are looking for time-dependent solutions of the 5-dimensional Einstein equations. A Friedmann equation governing the expansion rate of the brane is then found using Israel's junction conditions. Together with the usual energy conservation law, this equation completely describe the cosmological evolution on the brane. By construction, this approach fully takes into account the back-reaction due to the presence of the brane.

6.3.1 The five-dimensional Einstein equations

In this section we follow the approach of Binétruy *et al.* [18]. These authors considered a 5-dimensional space-time $(\mathcal{M}^5, G)$ with an action

$$S_5 = \frac{1}{2\kappa_5^2} \int \mathrm{d}^5x\sqrt{-G}\, R - \int \mathrm{d}^5x\sqrt{-G}\, \mathcal{L}_m, \tag{6.11}$$

where $1/2\kappa_5^2 = M_5^3/2$, and $\mathcal{L}_m$ is the Lagrangian density describing matter fields in the bulk and on the brane. The gravitational action is obtained from the dilaton-gravity action (2.54) for $D = 5$ by setting the dilaton to zero (see paragraph 2.5.2). The corresponding Einstein's equations are

$$\mathcal{G}_{MN} = R_{MN} - \frac{1}{2}\, G_{MN}R = \kappa_5^2 T_{MN}, \tag{6.12}$$

and we assume, that they completely describe the dynamics of the bulk.

To take into account the isotropy and homogeneity of our universe, we require $\mathcal{M}^5$ to contain a maximally symmetric 3-dimensional subspace $(\mathcal{M}^3, \gamma)$. This leads to the ansatz

$$\begin{aligned} \mathrm{d}s_5^2 &= G_{MN}\mathrm{d}x^M\mathrm{d}x^N \\ &= -n^2(t,y)\mathrm{d}t^2 + a^2(t,y)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j + b^2(t,y)\mathrm{d}y^2, \end{aligned} \tag{6.13}$$

where y is the coordinate of the extra-dimension. Here, we do not specify whether it is compact or not. Furthermore, if our universe is taken to be the hypersurface $\{y = 0\}$, the usual scale factor is simply given by $a(t, y = 0)$. Like in the RS model this metric is not factorizable. In some places we use the alternative (but completely equivalent) parametrization

$$\mathrm{d}s_5^2 = -\mathrm{e}^{2N(t,y)}\mathrm{d}t^2 + \mathrm{e}^{2R(t,y)}\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{e}^{2B(t,y)}\mathrm{d}y^2. \tag{6.14}$$

The RS metric (6.4) is recovered by setting $N(y) = R(y), B = 1, \gamma_{ij} = \delta_{ij}$, and we shall mostly adopt this form in Chap. 7.

The components of the Einstein tensor constructed with the metric (6.14) are

$$\begin{aligned}
\frac{1}{3}\,\mathcal{G}_{00} &= \dot{R}^2 + \dot{R}\dot{B} + \mathrm{e}^{2(N-B)}(-R'' - 2R'^2 + R'B') + \mathcal{K}\mathrm{e}^{2(N-R)},\\
\mathcal{G}_{0i} &= 0,\\
\mathcal{G}_{ij} &= \mathrm{e}^{2(R-N)}\gamma_{ij}(-2\ddot{R} - \ddot{B} - 3\dot{R}^2 - \dot{B}^2 + 2\dot{N}\dot{R} + \dot{N}\dot{B} - 2\dot{R}\dot{B})\\
&\quad + \mathrm{e}^{2(R-B)}\gamma_{ij}(N'' + 2R'' + N'^2 + 3R'^2 + 2N'R' - N'B' - 2R'B') - \mathcal{K}\gamma_{ij},\\
\frac{1}{3}\,\mathcal{G}_{04} &= -\dot{R}' - \dot{R}R' + N'\dot{R} + R'\dot{B},\\
\frac{1}{3}\,\mathcal{G}_{44} &= \mathrm{e}^{2(B-N)}(-\ddot{R} - 2\dot{R}^2 + \dot{N}\dot{R}) + N'R' + R'^2 - \mathcal{K}\mathrm{e}^{2(B-R)},
\end{aligned} \tag{6.15}$$

where a dot denotes the derivative with respect to t, and a prime that with respect to y. The constant $\mathcal{K}$ is the curvature of the subspace $\mathcal{M}^3$ with $\mathcal{K} = +1, 0, -1$. Later, in (7.12)-(7.17), we give Einstein's equations for a more general setup including the dilaton and a scalar field on the brane and potential terms for both of them.

The 5-dimensional energy-momentum tensor is decomposed into a bulk part T_B and a part describing matter fields on the brane[6],

$$T_{MN} = (T_B)_{MN} + \delta^\mu_M \delta^\nu_N \frac{\delta(y)}{b}\tau_{\mu\nu}, \tag{6.16}$$

where

$$\begin{aligned}
(T_B)^M{}_N &= \mathrm{diag}(-\rho_B, P_B, P_B, P_B, P_T),\\
\tau^\mu{}_\nu &= \mathrm{diag}(-\rho, P, P, P).
\end{aligned} \tag{6.17}$$

In this ansatz, one supposes that T_B is independent of y, and that $(T_B)_{04} = 0$, *i.e.* there is no energy flow along the fifth dimension.

Equating the components (6.15) of the Einstein tensor with their corresponding matter parts (6.17), one obtains a coupled system of non linear partial differential equations of second order. Remarkably, it is possible to find a first integral[7], provided that $-\rho_B = P_B = P_T$, *i.e.* if the energy-momentum tensor in the bulk is that of a cosmological constant $\Lambda = \kappa_5^2\rho_B$. In the notation $n \equiv \mathrm{e}^N, a \equiv \mathrm{e}^R, b \equiv \mathrm{e}^B$

[6] The factor $1/b$ is included to assure that $\int \mathrm{d}y\sqrt{G_{44}}\frac{\delta(y)}{b}\tau_{00} = \rho$ with $G_{44} = b^2$.

[7] See Ref. [18] for a more detailed discussion.

the integrated Einstein equations read

$$\left(\frac{\dot{a}}{na}\right)^2 = \frac{\kappa_5^2}{6}\rho_B + \left(\frac{a'}{ba}\right)^2 - \frac{\mathcal{K}}{a^2} + \frac{\mathcal{C}}{a^4}, \tag{6.18}$$

where $\mathcal{C}$ is an integration constant. This equation is valid for all values of y. Notice also that the presence of the brane has not yet been taken into account. If the bulk is stabilized, *i.e.* $\dot{b} = 0$, it possible to integrate Eq. (6.18) to find a global solution for $a(t,y)$ and $n(t,y)$. Here, we are only interested in evaluating Eq. (6.18) at the location of the brane, $y = 0$, in order to find an evolution equation for the scale factor $a_0 = a(t, y = 0)$. To do so, we need Israel's junctions conditions.

6.3.2 Junction conditions and the Friedmann equation of brane cosmology

In Eq. (6.18) the value of $a'(t, y = 0)$ is unknown, and therefore we cannot just evaluated it at $y = 0$ to find a Friedmann-like equation on the brane. However, we can assume that the bulk is Z_2-symmetric. Then, the metric (and thus a) is continuous at $y = 0$, and the scale factor satisfies $a(-y) = a(y)$ and $a'(-y) = -a'(y)$. One defines the jump of a' across $y = 0$ by

$$[a'] \equiv \lim_{y\to 0} \left(a'(t,y) - a'(t,-y)\right), \tag{6.19}$$

and hence in a Z_2-symmetric setup

$$a'(t, y = 0) = \frac{1}{2}[a']. \tag{6.20}$$

Israel's junction conditions state that this jump is non zero if there is a hypersurface with some energy content at $y = 0$. Here, this hypersurface is our universe brane, which affects the bulk geometry due to its matter and radiation content.

For the general form of the junction conditions, we refer to paragraph 3.2.6 and in particular to Eqs. (3.50) and (3.51). Here, we undertake a more 'pedestrian' approach: because of the jump in a', the second derivative a'', and thus Einstein equations (6.15), contain Dirac delta functions. To treat the regular and distributional parts separately, one writes

$$a'' = a''_{\rm reg} + \delta(y)[a'], \tag{6.21}$$

and similarly for n or, in the alternative notation, for the exponents R and N. Equating the distributional parts in Eqs. (6.15) with those in the energy-momentum tensor (6.16) leads to the conditions

$$\begin{aligned} [R'] &= -\frac{\kappa_5^2}{3}\mathrm{e}^{B_0}\rho \iff \frac{[a']}{a_0 b_0} = -\frac{\kappa_5^2}{3}\rho, \\ [N'] &= \frac{\kappa_5^2}{3}\mathrm{e}^{B_0}(3P + 2\rho) \iff \frac{[n']}{n_0 b_0} = \frac{\kappa_5^2}{3}(3P + 2\rho). \end{aligned} \tag{6.22}$$

These equations can be understood as the integral of the Einstein equations across the brane. The derivative a' is proportional to the extrinsic curvature. Equivalently, we could have obtained the Eqs. (6.22) via the general expression (3.51).

Let us now go back to Eq. (6.18). With a suitable change of the time coordinate t, for the fixed value $y = 0$, one can always obtain $n_0 = 1$. The new time coordinate then corresponds to cosmic time τ, and the dots will henceforth denote derivatives with respect to τ. Using Eq. (6.20), $[a'] = 2a_0'$, together with the first equation in (6.22), yields

$$H^2 \equiv \left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{\kappa_5^2}{6}\rho_B + \frac{\kappa_5^4}{36}\rho^2 - \frac{\mathcal{K}}{a_0^2} + \frac{\mathcal{C}}{a_0^4}. \tag{6.23}$$

This is a Friedmann-like equation for the expansion rate H of a 3-brane embedded in 5-dimensional bulk. Remarkably, the brane energy density ρ enters quadratically, in contrast to the usual Friedmann equation $H^2 \sim \rho$. The cosmological evolution also depends on a bulk cosmological constant $\Lambda = \kappa_5^2\rho_B$ and a term $\mathcal{C}/a_0^4$, where $\mathcal{C}$ is determined by initial conditions in the bulk. This term corresponds to radiation, though it is not physical radiation. The curvature term $\mathcal{K}/a_0^2$ is as in standard cosmology.

Eq. (6.23) is valid for any equation of state, $P = \omega\rho$, on the brane, since the only assumption we have made on $\tau_{\mu\nu}$ is that it represent a perfect fluid. The above equation is sufficient to study the cosmological evolution of a brane universe. There is no need to know the geometry outside the brane nor the evolution of the extra-dimension $b(t, y)$, provided that ρ_B and $\mathcal{C}$ are fixed.

From the Bianchi identities $\nabla_M \mathcal{G}^{MN} = 0$ one can derive an energy conservation equation. If $(T_B)_{04} = 0$ it takes the usual form

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{6.24}$$

If $(T_B)_{04} \neq 0$ there is an additional term describing the energy flow from the brane into the bulk and vice versa.

6.3.3 Solution of the Friedmann equation and recovering standard cosmology

The ρ^2 term in Eq. (6.23) leads to a cosmological evolution which is different from the standard one. In particular, in the early universe, the ρ^2 term is important. Strong constraints on this non standard behavior arise from the expansion rate during Nucleosynthesis. It has been shown in [19] that a pure $H^2 \sim \rho^2$ law does not satisfy Nucleosynthesis bounds. A possible solution, suggested in [18], is to decompose the energy density into a part due to real matter, ρ, and a part due to the brane tension, ρ_T, by replacing $\rho \to \rho + \rho_T$ in Eq. (6.23). One then obtains

$$H^2 = \frac{\kappa_5^2}{6}\rho_B + \frac{\kappa_5^4}{36}\rho_T^2 + \frac{\kappa_5^4}{18}\rho_T\rho + \frac{\kappa_5^4}{36}\rho^2 - \frac{\mathcal{K}}{a_0^2} + \frac{\mathcal{C}}{a_0^4}. \tag{6.25}$$

At late times, $\rho \ll \rho_T$ so that the ρ^2 contribution is negligible. Furthermore, the negative bulk energy density can be cancelled by the term in ρ_T^2 by imposing that

$$\frac{\kappa_5^2}{6}\rho_B + \frac{\kappa_5^4}{36}\rho_T^2 = 0, \tag{6.26}$$

which is nothing else but the RS fine-tuning condition (6.6). The fine-tuning (6.26) is the brane world version of the cosmological constant problem, and perturbations of this relation will be investigated in Chap. 7. Finally, we make the identification

$$\frac{\kappa_5^4}{18}\rho_T = \frac{8\pi G_4}{3}. \tag{6.27}$$

Thus, Newton's constant G_4 is determined by κ_5 and the brane tension. The tension needs to be positive, otherwise gravity on the brane would be repulsive. This causes some trouble for the first model of RS.

With these assumptions equation (6.25) takes the standard form

$$H^2 = \frac{8\pi G_4}{3}\rho - \frac{\mathcal{K}}{a_0^2} + \frac{\mathcal{C}}{a_0^4}, \tag{6.28}$$

of the 4-dimensional Friedmann equation (apart form the last term, which can be neglected in the matter dominated era).

It is interesting to compare Eq. (6.28) to the corresponding equation in mirage cosmology, Eq. (4.18)[8]. There, we have also found a term $1/a_0^4$, mimicking the behavior of radiation, which was due to the conserved energy E of the moving brane as well as to the black hole horizon r_H. Therefore, the parameter $\mathcal{C}$ can be related to those quantities. The terms $1/a_0^6, 1/a_0^8, 1/a_0^{10}$ found in mirage cosmology are absent here. In contrast, the ρ term allows to take into account real matter and real radiation on the brane, which source the cosmological expansion. For these reasons, and since it is not clear how to reproduce a $1/a_0^3$ term in mirage cosmology, it seems necessary to include the back-reaction. In the co-dimension 1 case, a detailed discussion on the the link between the mirage cosmology approach and the junction condition approach has been made in Ref. [144].

We conclude this section by giving a particular solution of the brane Friedmann equation (6.25). Consider the case where the fine-tuning condition (6.26) is satisfied and set $\mathcal{K} = \mathcal{C} = 0$. Furthermore, assume an equation of state $P = \omega\rho$, such that the energy conservation law on the brane, Eq. (6.24), can be integrated. Thus,

$$\rho = \rho_i \left(\frac{a_i}{a_0}\right)^{3(1+\omega)}, \tag{6.29}$$

where ρ_i and a_i are constants determined by the initial conditions (see also Eq. (1.34)). Substituting this expression into Eq. (6.25), one finds after integration,

$$a_0(\tau) = a_i \left[\kappa_5^2\rho_i \left(\frac{q^2}{72}\kappa_5^2\rho_T\tau^2 + \frac{q}{6}\tau\right)\right]^{1/q} \tag{6.30}$$

[8]In Chaps. 4 and 5 we have denoted the scale factor on the brane by a rather than a_0.

with $q = 3(1+\omega)$. Thus, the early universe evolution is characterized by an $a \sim \tau^{1/q}$ law, which in the case of radiation ($\omega = 1/3, q = 4$) gives $a \sim \tau^{1/4}$. At late times, the first term in Eq. (6.30) dominates, and the standard evolution $a \sim \tau^{2/q}$ is recovered. Notice that this crucially relies on the assumption of the fine-tuning (6.26).

Another interesting possibility arises, if the relation (6.26) is not quite zero, but slightly positive. Then, this term has the effect of a cosmological constant on the brane, which could drive late time acceleration.

6.4 Einstein's equations on the brane world

In the previous section we have shown how to derive an evolution equation on the brane from the 5-dimensional Einstein equations (6.12). Alternatively, one can derive equations for the 4-dimensional Einstein tensor on the brane by using the relations of Gauss, Codazzi and Mainardi (see paragraph 3.2.5). Those induced Einstein equations can be written mostly in terms of internal brane quantities. Since the fundamental gravity, however, is 5-dimensional, one expects corrections to the usual 4-dimensional Einstein equations. In this section we closely follow Ref. [142].

Consider a 3-brane $(\mathcal{M}^4, g)$ embedded in a 5-dimensional space-time $(\mathcal{M}^5, G)$. Locally the 5-dimensional metric can be decomposed according to Eq. (3.21)

$$G_{MN} = q_{MN} + n_M n_N, \tag{6.31}$$

where q_{MN} is the first fundamental form and n_M the unit normal one-form to the brane. We only consider the simplest case where the brane is fixed at $y = 0$, and so the tangent vectors are $e^M_\mu = \delta^M_\mu$. Since the brane is not moving, the unit normal vector n^M has a single non zero component in the y-direction. The relation (3.12) between the first fundamental form and the induced metric is

$$q^{MN} = g^{\mu\nu}\delta^M_\mu \delta^N_\nu. \tag{6.32}$$

The derivation of the induced Einstein equations is somewhat technical, and so we refer the reader to Ref. [142]. Essentially, it is performed using the differential geometry formalism presented in Sec. 3.2. Here, we just mention the important steps. According to the Gauss equation (3.35), the internal Riemann tensor on the brane can be written in terms of the 5-dimensional Riemann tensor and the second fundamental form K_{MN}. By contraction one obtains the 4-dimensional Einstein tensor $\widehat{\mathcal{G}}_{\mu\nu}$ as a function of $(R_{MNRS}, R_{MN}, R, K^2_{MN})$. The 5-dimensional Riemann tensor R_{MNRS} is then decomposed into (C_{MNRS}, R_{MN}, R), where C denotes the Weyl tensor. Furthermore, the Ricci tensor R_{MN} and the Riemann scalar R can be eliminated in favor of T_{MN} using the 5-dimensional Einstein equations (6.12). One obtains, schematically,

$$\widehat{\mathcal{G}}_{\mu\nu} \sim (T_{MN}, C_{MNRS}, K^2_{MN}). \tag{6.33}$$

So far, the recipe for the derivation is completely general. To proceed, we choose Gaussian normal coordinates[9] based on the brane,

$$\mathrm{d}s_5^2 = q_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2, \tag{6.34}$$

and write the 5-dimensional energy-momentum tensor as

$$T_{MN} = -\rho_B G_{MN} + \delta^\mu_M \delta^\nu_N \delta(y) S_{\mu\nu}, \tag{6.35}$$

where ρ_B denotes the energy density associated with a cosmological constant in the bulk, $\rho_B = \Lambda/\kappa_5^2$. The surface energy-momentum tensor,

$$S_{\mu\nu} = -\rho_T g_{\mu\nu} + \tau_{\mu\nu}, \tag{6.36}$$

contains the energy density ρ_T associated with the brane tension and $\tau_{\mu\nu}$, the usual 4-dimensional energy-momentum tensor of fluid matter. Notice that this corresponds to the splitting $\rho \to \rho + \rho_T$ in the previous section. Eqs. (6.35) and (6.36) are now inserted into Eq. (6.33).

Finally we assume that the bulk is Z_2-symmetric. Then, one can use Israel's junction conditions (3.51) to express K_{MN} in Eq. (6.33) in terms of $S_{\mu\nu}$. The final form for the Einstein equations in a 3-brane world reads

$$\widehat{\mathcal{G}}_{\mu\nu} = \widehat{R}_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\widehat{R} = -\Lambda_4 g_{\mu\nu} + 8\pi G_4 \tau_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} - E_{\mu\nu}, \tag{6.37}$$

where

$$\begin{aligned}
\Lambda_4 &= \frac{1}{2}\kappa_5^2 \left(\rho_B + \frac{1}{6}\kappa_5^2 \rho_T^2\right), \\
G_4 &= \frac{\kappa_5^4}{48\pi}\rho_T, \\
\pi_{\mu\nu} &= -\frac{1}{4}\tau_{\mu\alpha}\tau^\alpha_\nu + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}g_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24}g_{\mu\nu}\tau^2, \quad \tau = \tau^\alpha_\alpha, \\
E_{\mu\nu} &= C^A{}_{BRS} n_A n^R q^B_\mu q^S_\nu.
\end{aligned} \tag{6.38}$$

The quantity Λ_4 is the effective 4-dimensional cosmological constant on the brane, which is a combination of the negative bulk energy density ρ_B and the energy density corresponding to the brane tension, ρ_T. The case $\Lambda_4 = 0$ corresponds to the RS fine-tuning (6.6) in a cosmological context. As in the section before, it is found that Newton's constant G_4 is given by κ_5 and ρ_T. In the above Einstein equations, the matter source is represented by $\tau_{\mu\nu}$ and $\pi_{\mu\nu}$. The latter is quadratic in $\tau_{\mu\nu}$, which traces back to the Gauss equation (3.35): there, the extrinsic curvature tensor enters quadratically, which via the junction conditions translates into terms $S^2_{\mu\nu}$ and finally $\tau^2_{\mu\nu}$. As a consequence for cosmology, the unusual ρ^2 term shows up in the Friedmann equation (6.23). Furthermore, there is a term $E_{\mu\nu}$, which is a projection of the 5-dimensional Weyl tensor onto the

[9] For a definition see Sec. 3.2

brane, and which cannot be determined from internal brane quantities alone. The Weyl tensor represents 5-dimensional gravitational waves propagating in the bulk, and $E_{\mu\nu}$ corresponds to their energy-momentum deposited on the brane. In a cosmological setting, this contribution shows up as $\mathcal{C}/a_0^4$ in the Friedmann equation[10].

In the Einstein equations, $E_{\mu\nu}$ must be evaluated either at $y = +0$ or $y = -0$, rather than on the brane itself. Because of that, the equations on the brane are not a closed system. For certain issues, for example in perturbation theory, one has to resort to the full 5-dimensional dynamics.

Both unusual terms, $\pi_{\mu\nu}$ and $E_{\mu\nu}$, play a role in the early (brane) universe, but can be neglected today. Hence, in a low energy limit, the usual Einstein equations are recovered.

[10]The Weyl tensor of an AdS_5 background is zero, and so $E_{\mu\nu}$ or $\mathcal{C}$, respectively, vanish. Notice also that $\mathcal{C} \sim r_H = 0$ in AdS_5 .

Chapter 7

Dynamical instabilities of the Randall-Sundrum model (article)

This chapter consists of the article 'Dynamical instabilities of the Randall-Sundrum model', published in Phys.Rev.D64.(2001), see Ref. [21].

It is also available under http://lanl.arXiv.org/abs/hep-th/0102144.

In this article, we work in units $2\kappa_5^2 = 1$. The fine-tuning condition (6.26) then becomes Eq. (7.28), where we have expressed it in terms of the 5-dimensional cosmological constant Λ rather than ρ_B. In this chapter, ϕ denotes the dilaton, as we need the variable Φ for a Bardeen-like potential.

Dynamical instabilities of the Randall-Sundrum model

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We derive dynamical equations to describe a single 3-brane containing fluid matter and a scalar field coupling to the dilaton and the gravitational field in a five-dimensional bulk. First, we show that a scalar field or an arbitrary fluid on the brane cannot evolve to cancel the cosmological constant in the bulk. Then we show that the Randall–Sundrum model is unstable under small deviations from the fine–tuning between the brane tension and the bulk cosmological constant and even under homogeneous gravitational perturbations. Implications for brane world cosmologies are discussed.

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7.1 Introduction

Until now, string theories are the most promising fundamental quantum theories at hand which include gravity. Open strings carry gauge charges and end on so-called Dp-branes, $(p+1)$–dimensional hypersurfaces of the full space-time. Correspondingly, gauge fields may propagate only on the $(p+1)$–dimensional brane, and only modes associated with closed strings, such as the graviton, the dilaton and the axion, live in the full space-time [125]. Superstring theories and especially M theory suggest that the observable universe is a $(3+1)$–dimensional hypersurface, a 3–brane, in a 10 or 11-dimensional space-time. This fundamental space-time could be a product of a four–dimensional Lorentz manifold with an n–dimensional compact space of volume V_n (n being the number of extra–dimensions). Then, the relation between the $(4+n)$–dimensional fundamental Planck mass M_{4+n} and the effective four–dimensional Planck mass $M_4 \equiv \sqrt{1/(8\pi G_4)} \simeq 2.4 \times 10^{18}$ GeV is

$$M_4^2 = M_{4+n}^{2+n} V_n. \tag{7.1}$$

If some of the extra dimensions are much larger than the fundamental Planck scale, M_{4+n} is much smaller than M_4 and may even be close to the electroweak scale, thereby relieving the long-standing hierarchy problem [12, 9]. For example, if one allows for two 'large' extra dimensions of the order of 1 mm, one obtains a fundamental Planck mass of 1 TeV. However, a new hierarchy between the electroweak scale and the mass scale associated with the compactification volume, ${V_n}^{-1/n}$ is introduced.

Clearly, this idea is very interesting from the point of view of bringing together fundamental theoretical high energy physics and experiments, which have been diverging more and more since the advent of string theory. While the four–dimensionality of gauge interactions has been tested down to scales of about $1/200\,\mathrm{GeV}^{-1} \simeq 10^{-15}\,\mathrm{mm}$, Newton's law is experimentally confirmed only above 1 mm. Therefore, 'large' extra dimensions are not excluded and should be tested in the near future by refined microgravity experiments [103, 79]. The fundamental string scale might in principle be accessible to the CERN Large Hadron Collider (LHC) [110, 10, 69, 134].

In the past, it was commonly assumed that the fundamental space-time is factorizable, and that the extra–dimensional space is compact. Recently, Randall and Sundrum [127] proposed a five–dimensional model, in which the metric on the 3–brane is multiplied by an exponentially decreasing 'warp' factor such that transverse lengths become small already at short distances along the fifth dimension. This idea allows for a non compact extra–dimension without getting into conflict with observational facts. In this scenario the brane is embedded in an anti–de Sitter space, and a fine–tuning relation

$$\Lambda = -\frac{\kappa_5^2}{6} V^2, \tag{7.2}$$

between the brane tension V and the negative cosmological constant in the bulk Λ has to be satisfied. Here, κ_5 is related to the five–dimensional Newton constant by $\kappa_5^2 = 6\pi^2 G_5 = M_5^{-3}$. Randall and Sundrum also proposed a model with two branes of opposite tension which provides an elegant way to relieve both hierarchy problems mentioned above [128]. However, also this model requires the fine–tuning (7.2). Here, we will only consider the case of a single brane.

The main unattractive feature of the Randall–Sundrum (RS) model is the fine-tuning condition (7.2). Both from the particle physics and the cosmological point of view this relation between two a priori independent quantities appears unlikely. One would like to put it on a physical basis, such as a fundamental principle, or explain it due to some dynamical process.

The purpose of this paper is to point out the cosmological problems associated with the fine–tuning condition (7.2). The outline of the paper is as follows: In Sec. 7.2, we derive dynamical equations describing the gravitational field and the dilaton in the bulk coupling to fluid matter and a scalar field on the brane. These equations allow for a dynamical generalization of the RS model, which is a special static solution of our equations with vanishing dilaton. Our equations also provide a starting point for further studies of various issues in cosmology, for example inflation. In Sec. 7.3 we discuss a cosmological version of the RS model and show that the fine–tuning condition (7.2) cannot be stabilized by an arbitrary scalar field or fluid on the brane. In Sec. 7.4 we discuss linear perturbations of the static RS model and derive gauge invariant perturbation equations from our general setup. We prove that the full RS space-time is unstable against homogeneous processes on the brane such as cosmological phase transitions: The solutions run quadratically fast away from the static RS space-time. This instability reminds that of the static homogeneous and isotropic Einstein universe [59]. In linear perturbation theory we also find a mode which represents an instability linear in time. In the last section we present our results and the conclusions.

7.2 Equations of motion

In this section we derive the equations of motion. For generality and for future work we have included the dilaton, although it does not play a role in the present discussion of RS stability. Works on dilaton gravity and the brane world have also been done by the authors of Refs. [109] and [108].

7.2.1 General case

We consider a five–dimensional space-time with metric G_{MN} parameterized by coordinates $(x^M) = (x^\mu, y)$, where $M = 0,1,2,3,4$ and $\mu = 0,1,2,3$, with a 3–brane fixed at $y = 0$. We use units in which $2\kappa_5^2 = 1$. In the string frame the action is

$$\begin{aligned} S_5 = &\int \mathrm{d}^5x\sqrt{-G}\,\mathrm{e}^{-2\phi}\left(R + 4(\nabla_M\phi)(\nabla_N\phi)G^{MN} - \Lambda(\phi)\right) \\ &- \int \mathrm{d}^4x\sqrt{-g}\,\mathrm{e}^{-2\phi}\left(\frac{1}{2}(\widehat{\nabla}_\mu\varphi)(\widehat{\nabla}_\nu\varphi)g^{\mu\nu} + V(\varphi) + \mathcal{L}\right), \end{aligned} \tag{7.3}$$

which describes the coupling of the dilaton ϕ to gravity, as well as to a scalar field φ with potential $V(\varphi)$ and to a fluid with Lagrangian $\mathcal{L}$ on the brane. The graviton, the dilaton and the 'bulk potential' $\Lambda(\phi)$ live in the five-dimensional bulk space-time, whereas the fluid and the scalar field are confined to the brane. The induced four-dimensional metric is[1]

$$g_{\mu\nu} = \delta_\mu^M\delta_\nu^N G_{MN}(y=0). \tag{7.4}$$

The action in the Einstein–frame is obtained by the conformal transformation

$$G_{MN} \to \mathrm{e}^{-\frac{4\phi}{D-2}}G_{MN} \tag{7.5}$$

with $D = 5$. We find

$$\begin{aligned} S_5^e = &\int \mathrm{d}^5x\sqrt{-G}\left(R - \frac{4}{3}(\nabla_M\phi)(\nabla_N\phi)G^{MN} - \mathrm{e}^{(4/3)\phi}\Lambda(\phi)\right) \\ &- \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}\,\mathrm{e}^{-(2/3)\phi}(\widehat{\nabla}_\mu\varphi)(\widehat{\nabla}_\nu\varphi)g^{\mu\nu} + \mathrm{e}^{(2/3)\phi}\left(V(\varphi) + \mathcal{L}\right)\right), \end{aligned} \tag{7.6}$$

[1] Where confusion could arise, we use a hat to denote four-dimensional quantities.

where G_{MN} now denotes the metric tensor in the Einstein–frame, and R and ∇ are constructed from this metric. The equation of motion for the dilaton is obtained by varying this action with respect to ϕ. We find

$$\begin{aligned}&\frac{8}{3}\nabla^2\phi - \frac{4}{3}\,\mathrm{e}^{(4/3)\phi}\Lambda(\phi) - \mathrm{e}^{(4/3)\phi}\frac{\partial\Lambda(\phi)}{\partial\phi}\\ &+\frac{\sqrt{-g}}{\sqrt{-G}}\delta(y)\left(\frac{1}{3}\,\mathrm{e}^{-(2/3)\phi}(\widehat{\nabla}\varphi)^2 - \frac{2}{3}\,\mathrm{e}^{(2/3)\phi}(V(\varphi)+\mathcal{L})\right) = 0.\end{aligned} \tag{7.7}$$

Similarly, the equation for φ is

$$\mathrm{e}^{-(2/3)\phi}(\widehat{\nabla}^2\varphi) - \mathrm{e}^{(2/3)\phi}\frac{\partial V(\varphi)}{\partial\varphi} = 0. \tag{7.8}$$

Finally, the 5-dimensional Einstein tensor is

$$\begin{aligned}\mathcal{G}_{MN} = &\frac{4}{3}\left((\nabla_M\phi)(\nabla_N\phi) - \frac{1}{2}\,G_{MN}(\nabla\phi)^2\right) - \frac{1}{2}\,G_{MN}\mathrm{e}^{(4/3)\phi}\Lambda(\phi)\\ &-\frac{\sqrt{-g}}{\sqrt{-G}}\delta(y)\delta^\mu_M\delta^\nu_N\Bigg(- \mathrm{e}^{-(2/3)\phi}\frac{1}{2}\left((\widehat{\nabla}_\mu\varphi)(\widehat{\nabla}_\nu\varphi) - \frac{1}{2}\,g_{\mu\nu}(\widehat{\nabla}\varphi)^2\right)\\ &+\frac{1}{2}\,g_{\mu\nu}\mathrm{e}^{(2/3)\phi}V(\varphi) - \frac{1}{2}\,\mathrm{e}^{(2/3)\phi}\tau_{\mu\nu}\Bigg),\end{aligned} \tag{7.9}$$

where $\tau_{\mu\nu}$ is the energy-momentum tensor of the fluid on the brane. As we are interested in cosmological solutions, we require the 3–brane to be homogeneous and isotropic and make the ansatz

$$\mathrm{d}s_5^2 = -\mathrm{e}^{2N(t,y)}\mathrm{d}t^2 + \mathrm{e}^{2R(t,y)}\mathrm{d}\vec{x}^2 + \mathrm{e}^{2B(t,y)}\mathrm{d}y^2, \tag{7.10}$$

where the ordinary spatial dimensions are assumed to be flat. Note that this metric is not factorizable as the scale factor on the brane $\mathrm{e}^{R(t,y)}$ and the lapse function $\mathrm{e}^{N(t,y)}$ depend on time as well as on the fifth dimension. The factor $\mathrm{e}^{B(t,y)}$ is a modulus field. The energy–momentum tensor of a homogeneous and isotropic fluid, representing matter in the universe, is

$$(\tau^\mu{}_\nu(t)) = \mathrm{diag}(-\rho(t), P(t), P(t), P(t)), \tag{7.11}$$

and for the dilaton and the brane scalar field we shall assume $\phi = \phi(t,y), \varphi = \varphi(t)$. Finally, the Lagrangian density of the fluid, $\mathcal{L}$, is given by its free energy density F (see Ref. [153]).

With these assumptions the equations of motion take the form below. An overdot and a prime refer to the derivatives with respect to t and y, and quantities on the brane carry a subscript zero, for example $N_0 \equiv N(t, y = 0)$.

The equation of motion for the dilation is

$$\begin{aligned}&\frac{8}{3}\,\mathrm{e}^{-2N}(\ddot\phi-\dot\phi\dot N+3\dot\phi\dot R+\dot\phi\dot B)-\frac{8}{3}\,\mathrm{e}^{-2B}(\phi''+\phi'N'+3\phi'R'-\phi'B')\\&+\frac{4}{3}\,\mathrm{e}^{(4/3)\phi}\Lambda(\phi)+\mathrm{e}^{(4/3)\phi}\frac{\partial\Lambda(\phi)}{\partial\phi}\\&+\delta(y)\mathrm{e}^{-B}\left(\frac{1}{3}\,\mathrm{e}^{-(2/3)\phi}\mathrm{e}^{-2N}\dot\varphi^2+\frac{2}{3}\,\mathrm{e}^{(2/3)\phi}(V(\varphi)+F)\right)=0.\end{aligned}\tag{7.12}$$

The equation of motion for the scalar field on the brane is

$$\mathrm{e}^{-2N_0}(\ddot\varphi-\dot\varphi\dot N_0+3\dot\varphi\dot R_0)+\mathrm{e}^{(4/3)\phi_0}\frac{\partial V(\varphi)}{\partial\varphi}=0.\tag{7.13}$$

The 00-component of the 5-dimensional Einstein equations is

$$\begin{aligned}&3\mathrm{e}^{-2N}\left(\dot R^2+\dot R\dot B-\frac{2}{9}\dot\phi^2\right)+3\mathrm{e}^{-2B}\left(-R''-2R'^2+R'B'-\frac{2}{9}\phi'^2\right)\\&-\frac{1}{2}\,\mathrm{e}^{(4/3)\phi}\Lambda(\phi)-\delta(y)\mathrm{e}^{-B}\left(\frac{1}{4}\,\mathrm{e}^{-(2/3)\phi}\mathrm{e}^{-2N}\dot\varphi^2+\frac{1}{2}\,\mathrm{e}^{(2/3)\phi}(V(\varphi)+\rho)\right)=0\;.\end{aligned}\tag{7.14}$$

The 11-component is

$$\begin{aligned}&\mathrm{e}^{-2N}\left(-2\ddot R-\ddot B-3\dot R^2-\dot B^2+2\dot N\dot R+\dot N\dot B-2\dot R\dot B-\frac{2}{3}\dot\phi^2\right)\\&+\mathrm{e}^{-2B}\left(N''+2R''+N'^2+3R'^2+2N'R'-N'B'-2R'B'+\frac{2}{3}\phi'^2\right)\\&+\frac{1}{2}\,\mathrm{e}^{(4/3)\phi}\Lambda(\phi)-\delta(y)\mathrm{e}^{-B}\left(\frac{1}{4}\,\mathrm{e}^{-(2/3)\phi}\mathrm{e}^{-2N}\dot\varphi^2+\frac{1}{2}\,\mathrm{e}^{(2/3)\phi}(P-V(\varphi))\right)=0\;.\end{aligned}\tag{7.15}$$

The 04-component is

$$\dot R'+\dot RR'-N'\dot R-R'\dot B+\frac{4}{9}\dot\phi\phi'=0\;.\tag{7.16}$$

The 44-component is

$$\begin{aligned}&3\mathrm{e}^{-2N}\left(-\ddot R-2\dot R^2+\dot N\dot R-\frac{2}{9}\dot\phi^2\right)+3\mathrm{e}^{-2B}\left(N'R'+R'^2-\frac{2}{9}\phi'^2\right)\\&+\frac{1}{2}\,\mathrm{e}^{(4/3)\phi}\Lambda(\phi)=0\;.\end{aligned}\tag{7.17}$$

Finally, for the fluid we assume an equation of state of the form

$$P=P(\rho).\tag{7.18}$$

In order to have a well defined geometry, the metric has to be continuous across $y=0$. However, first derivatives with respect to y may not be continuous at $y = 0$, and second derivatives may contain delta functions. Such distributional parts can be treated separately by writing

$$f'' = f''_{\rm reg} + \delta(y)[f'] \ , \tag{7.19}$$

where

$$[f'] \equiv \lim_{y\to 0} \left(f'(y) - f'(-y)\right) \tag{7.20}$$

is the jump of f' across $y = 0$, and $f''_{\rm reg}$ is the part which is regular at $y = 0$. By matching the delta functions from the second derivatives of ϕ, N and R with those in equations (7.12), (7.14) and (7.15), one obtains the junction conditions

$$[\phi'] = \frac{1}{8}\,{\rm e}^{-(2/3)\phi_0}{\rm e}^{B_0-2N_0}\dot{\varphi}^2 + \frac{1}{4}\,{\rm e}^{(2/3)\phi_0}{\rm e}^{B_0}(V(\varphi) + F) \ , \tag{7.21}$$

$$[N'] = \frac{5}{12}\,{\rm e}^{-(2/3)\phi_0}{\rm e}^{B_0-2N_0}\dot{\varphi}^2 + \frac{1}{6}\,{\rm e}^{(2/3)\phi_0}{\rm e}^{B_0}(3P + 2\rho - V(\varphi)) \ , \tag{7.22}$$

$$[R'] = -\frac{1}{12}\,{\rm e}^{-(2/3)\phi_0}{\rm e}^{B_0-2N_0}\dot{\varphi}^2 - \frac{1}{6}\,{\rm e}^{(2/3)\phi_0}{\rm e}^{B_0}(V(\varphi) + \rho) \ . \tag{7.23}$$

Equations (7.22) and (7.23) are equivalent to Israel's junction conditions [82]. Our equations agree with those found by other authors in special cases, see *e.g.* Refs. [19] and [85].

In the remainder of this paper, we assume Z_2 symmetry. Furthermore, we neglect the dilaton and consider $\Lambda(\phi)$ to be a constant.

7.2.2 Special case: The Randall–Sundrum model

The RS model is a special static solution of the equations derived in the previous section with $N(y) = R(y), B = 0$, when Λ is taken to be a pure cosmological constant, and V represents a constant brane tension. All other fields are set to zero. The RS metric is

$$ds_5^2 = {\rm e}^{2\alpha|y|}(-{\rm d}t^2 + {\rm d}\vec{x}^2) + {\rm d}y^2 \ . \tag{7.24}$$

Our equations of motion then reduce to

$$6R'^2 = -\frac{\Lambda}{2} \ , \tag{7.25}$$

$$3R'' = -\delta(y)\frac{V}{2} \ . \tag{7.26}$$

Equation (7.25) can now be solved by

$$R(y) = -\sqrt{\frac{-\Lambda}{12}}|y| \equiv \alpha|y| \ , \tag{7.27}$$

which respects Z_2 symmetry and leads to an exponentially decreasing 'warp factor'. To satisfy simultaneously equation (7.26), one must fine-tune the brane tension and the (negative) bulk cosmological constant

$$\Lambda + \frac{V^2}{12} = 0 \ . \tag{7.28}$$

This is the RS solution. A priori, Λ and V are independent constants, and there is no reason that such a relation should hold. However, in a realistic time-dependent cosmological model this relation must be satisfied in order to recover the usual Friedmann equation for a fluid with $\rho \ll V$ see Ref. [43, 44, 18]. In the next section we study whether Eq. (7.28) can be obtained by some dynamics on the brane.

7.3 A dynamical brane

We first consider a dynamical scalar field on the brane. The fine-tuning condition (7.28) corresponds to the requirement that the negative bulk cosmological constant Λ can be cancelled by the brane tension V which we try to identify with the potential energy of the scalar field φ. If, starting with some initial conditions on φ and $\dot{\varphi}$, the evolution of the system would stabilize at $\Lambda + \frac{V^2}{12} = 0$, the cancellation could be accomplished dynamically. If this would be the case for a 'large class' of initial conditions, the RS solution (7.27) would be an attractor of the system.

We start from Eqs. (7.13)-(7.17) for the case of a vanishing dilaton. Taking the 'mean value' of the 44 equation across $y = 0$, inserting the junction conditions (7.22), (7.23) and taking into account Z_2 symmetry, one obtains (see Ref. [19])

$$\ddot{R}_0 + 2\dot{R}_0^2 = -\frac{1}{144}\rho_b(\rho_b + 3P_b) + \frac{\Lambda}{6}, \tag{7.29}$$

where $\rho_b = \rho + \rho_\varphi$ and $P_b = P + P_\varphi$ are the total energy density and the total pressure on the brane due to the fluid and the scalar field. In this section the overdot denotes the derivative with respect to the time coordinate τ given by $\mathrm{d}\tau = \mathrm{e}^{N_0(t)}\mathrm{d}t$. Using the energy conservation equation on the brane,

$$\dot{\rho}_b = -3\dot{R}_0(\rho_b + P_b), \tag{7.30}$$

one can eliminate the pressure and integrate Eq. (7.29) to obtain a 'Friedmann' equation for the expansion of the brane (see Refs. [43, 44, 18])

$$H^2 = \frac{1}{12}\Lambda + \frac{1}{144}{\rho_b}^2 + \frac{\mathcal{C}}{a_0^4}, \tag{7.31}$$

where $a_0(t) \equiv \mathrm{e}^{R(t,y=0)}$ denotes the scale factor on the brane, $H = \dot{a}_0/a_0 = \dot{R}_0$, and $\mathcal{C}$ is an integration constant. If the dilaton vanishes, Eq. (7.13) becomes the ordinary equation of motion for a scalar field

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0 \tag{7.32}$$

with an energy density and pressure

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{7.33}$$

$$P_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi). \tag{7.34}$$

We now assume that the energy density of the scalar field dominates any other component on the brane, that is $\rho_\varphi \gg \rho$. This may be the case in the early universe. Later in this section we will see that this assumption does not affect our result. In the same sense we neglect the radiation term, so that Eq. (7.31) reduces to

$$H = +\sqrt{\frac{1}{12}\left(\Lambda + \frac{{\rho_\varphi}^2}{12}\right)}. \tag{7.35}$$

The positive sign corresponds to an expanding brane. The question of whether the system evolves towards $\Lambda + \frac{V^2}{12} = 0$ is now translated into the question of whether the Hubble parameter vanishes at some time τ_1. From Eqs. (7.32) and (7.35) together with Eq. (7.33) one finds

$$\dot{H} = -\frac{1}{48}\rho_\varphi \dot{\varphi}^2, \tag{7.36}$$

which is always negative. (The case $\dot{\varphi}(\tau_1) = 0$ simultaneously with $H(\tau_1) = 0$ will be treated separately.) Starting with an expanding universe, $H > 0$, this implies that H is indeed decreasing and $H = 0$ may well be obtained within finite or infinite time depending on the details of the potential $V(\varphi)$. However, at τ_1 the scale factor has reached a maximum ($\ddot{a}_0(\tau_1) = a_0(\tau_1)\dot{H}(\tau_1) < 0$) and, after a momentary cancellation of Λ with ${\rho_\varphi}^2$, H changes sign and the brane begins to contract with

$$H = -\sqrt{\frac{1}{12}\left(\Lambda + \frac{{\rho_\varphi}^2}{12}\right)}\ . \tag{7.37}$$

In order for H to stop evolving at τ_1 when the RS condition $\Lambda + {\rho_\varphi}^2/12 = 0$ is satisfied, we need $\frac{d^n}{d\tau^n}H(\tau_1) = 0$ for all $n \geq 0$, which implies $\frac{d^n}{d\tau^n}\rho_\varphi = 0$ and also $\frac{d^n}{d\tau^n}\varphi(\tau_1) = 0$ for all $n \geq 1$. Therefore, the scalar field has to be constant with value $\varphi_1 \equiv \varphi(\tau_1)$ and $V(\varphi_1) = \sqrt{-12\Lambda}$. But this is only possible if $V(\varphi_1)$ is a minimum of the potential, and we have to put φ into this minimum with zero initial velocity from the start. This of course corresponds to the trivial static fine-tuned RS solution.

We conclude that the fine-tuning condition (7.28) cannot be obtained by such a mechanism. Note that our arguments have been entirely general and we have thus shown that the fine-tuning problem cannot be resolved by an arbitrary brane scalar field.

To illustrate the dynamics, we consider the potential $V(\varphi) = \frac{1}{2}m^2\varphi^2$. Eq. (7.35) then takes the form

$$H^2 = \frac{1}{12}\Lambda + \frac{1}{144}\left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}m^2\varphi^2\right)^2\ . \tag{7.38}$$

It is convenient to use dimensionless variables x, y, z and η related to φ, $\dot{\varphi}$, H

and τ by

$$\varphi \equiv \sqrt{\frac{24}{m}}x, \quad \dot{\varphi} \equiv \sqrt{24m}y, \quad H \equiv mz, \quad \tau \equiv \frac{1}{m}\eta. \tag{7.39}$$

Equations (7.32) and (7.38) are equivalent to a two–dimensional dynamical system in the phase space (x, y) with

$$x' = y, \quad y' = -x - 3zy, \tag{7.40}$$

with the constraint equation

$$z^2 = -K + (x^2 + y^2)^2. \tag{7.41}$$

The prime denotes the derivative with respect to the 'time parameter' η and $K \equiv -\frac{1}{12m^2}\Lambda$. In Fig. 7.1 two typical trajectories found by numerical solution of the system (7.40)-(7.41) are shown in the phase space (x, y). For large initial y, the damping term first dominates and lowers y until the potential term becomes comparable. Then, the system evolves towards the minimum of the potential until the curve hits the circle $x^2 + y^2 = \sqrt{K}$ (after finite time), where the damping term changes sign, and the trajectories move away nearly in the y direction. In Fig. 7.2 the corresponding evolution of the Hubble parameter z is shown.

In ordinary four–dimensional cosmology there exists a 'no–go theorem' due to Weinberg [162], which states that the cosmological constant cannot be cancelled by a scalar field. The argument is based on symmetries of the Lagrangian. In brane cosmology it is known [142] that for a 3–brane, embedded in a five–dimensional space-time, Einstein equations on the brane are the same as the usual four–dimensional Einstein equations, apart from two additional terms: a term $\pi_{\mu\nu}$ which is quadratic in the matter energy–momentum tensor $\tau_{\mu\nu}$ on the brane[2],

$$\pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\alpha}\tau_\nu{}^\alpha + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}g_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24}g_{\mu\nu}\tau^2, \tag{7.42}$$

and a term $E_{\mu\nu}$, which is the projection of the five–dimensional Weyl–tensor, $C^A{}_{BCD}$, onto the brane

$$E_{\mu\nu} \equiv C^A{}_{BCD} n_A n^C q_\mu{}^B q_\nu{}^D, \tag{7.43}$$

where n^A is the normal vector to the brane, q_{MN} is the first fundamental form, and τ is the trace of the energy momentum tensor. Being constructed purely from $\tau_{\mu\nu}$, the quantity $\pi_{\mu\nu}$ does not introduce additional dynamical degrees of freedom. It just contributes the term $\rho_\varphi{}^2$ to the 'Friedmann' equation (7.31). It is also clear that $E_{\mu\nu}$, which is traceless, cannot cancel the cosmological constant on the brane. However, since the effective Einstein equations on the brane cannot be derived from a Lagrangian, and since $E_{\mu\nu}$ contains additional information from the bulk, it is not evident that Weinberg's theorem holds in our case.

[2]Notice that the quantity $\tau_{\mu\nu}$ does not include the energy contribution due to the brane tension, see Eq. (6.36)

Figure 7.1: Two trajectories in the phase space (x, y) which represent typical solutions of the system (7.40)-(7.41) for $K = 1$. The trajectory on the left (dotted, red), starting with an initial condition $x_{\rm in} = -3, y_{\rm in} = 2$, winds towards the circle $x^2 + y^2 = \sqrt{K}$, which corresponds to the condition $z = H = 0$. After reaching the circle, the solution moves away showing that it is not an attractor. The trajectory on the right (solid) with initial conditions $x_{\rm in} = 2, y_{\rm in} = 2$ shows a similar behavior. It takes much longer to pass the region around the kinks than to trace out the remaining parts of the trajectories.

More generally, our 'no-go' result also holds for any matter obeying an equation of state $P = \omega\rho$ when $\omega > -1$. This can be seen in a similar way: initially the Hubble parameter is

$$H = +\sqrt{\frac{1}{12}\left(\Lambda + \frac{\rho^2}{12}\right)}. \tag{7.44}$$

Using the energy conservation equation

$$\dot{\rho} = -3H(1+\omega)\rho, \tag{7.45}$$

one finds

$$\dot{H} = -\frac{1}{48}(1+\omega)\rho^2, \tag{7.46}$$

and hence $\dot{H} < 0$ as long as the condition $\omega > -1$ (or $P > -\rho$) is satisfied. To relate our finding to previous results [43, 44, 18, 107], let us note that Eqs. (7.44)

Figure 7.2: The time evolution of the dimensionless Hubble parameter z for the two trajectories shown in Fig. 1.

and (7.46) imply the following condition for inflation on the brane

$$\frac{\ddot{a}_0}{a_0} = \dot{H} + H^2 = \frac{\Lambda}{12} - \frac{2+3\omega}{144}\rho^2 > 0. \tag{7.47}$$

For a brane energy density given by the brane RS tension $V = \sqrt{-12\Lambda}$ and an additional component indicated by a subscript $_f$, so that $\rho = V + \rho_f$ and $P = -V + P_f$ this gives

$$\frac{\ddot{a}_0}{a_0} = -[V(1+3\omega_f) + \rho_f(2+3\omega_f)]\frac{\rho_f}{144} > 0 \ , \tag{7.48}$$

which coincides with Eq. (8) of Ref [107]. If the RS term dominates, $V \gg \rho_f$ we obtain the usual strong energy condition for inflation, $1+3\omega_f < 0$, but if $V \ll \rho_f$ the condition is stronger, namely $2+3\omega_f < 0$.

As in the case of the scalar field, the brane starts to contract as soon as $H = 0$ is reached. We have thus shown that a relation such as Eq. (7.28) cannot be realized in a cosmological setting with matter satisfying $\omega > -1$.

After this section, in which we adopted the viewpoint of the brane, we now come back to the full five-dimensional space-time to investigate the stability of the RS model.

7.4 Gauge Invariant Perturbation equations

We formally prove that the five-dimensional RS space-time is unstable under small homogeneous perturbations of the brane tension.

7.4.1 Perturbations of the Randall–Sundrum model

The equations of motion derived in Sec. 7.2 provide with $N(t,y), R(t,y)$, and $B(t,y)$ a dynamical generalization of the RS model. We consider Λ and V to be constant and set the dilaton, the scalar field on the brane and the energy-momentum tensor of the fluid to zero. Equations (7.14)-(7.17) now reduce to

$$\begin{aligned} 00: \quad & 3\mathrm{e}^{-2N}(\dot{R}^2+\dot{R}\dot{B})+3\mathrm{e}^{-2B}(-R''-2R'^2+R'B') \\ - \quad & \frac{1}{2}\Lambda-\delta(y)\frac{1}{2}\,\mathrm{e}^{-B}V=0\ , \end{aligned} \tag{7.49}$$

$$\begin{aligned} 11: \quad & \mathrm{e}^{-2N}(-2\ddot{R}-\ddot{B}-3\dot{R}^2-\dot{B}^2+2\dot{N}\dot{R}+\dot{N}\dot{B}-2\dot{R}\dot{B}) \\ + \quad & \mathrm{e}^{-2B}(N''+2R''+N'^2+3R'^2+2N'R'-N'B'-2R'B') \\ + \quad & \frac{1}{2}\Lambda+\delta(y)\frac{1}{2}\,\mathrm{e}^{-B}V=0\ , \end{aligned} \tag{7.50}$$

$$04: \quad \dot{R}'+\dot{R}R'-N'\dot{R}-R'\dot{B}=0\ , \tag{7.51}$$

$$44: \quad 3\mathrm{e}^{-2N}(-\ddot{R}-2\dot{R}^2+\dot{N}\dot{R})+3\mathrm{e}^{-2B}(N'R'+R'^2)+\frac{1}{2}\Lambda=0 \tag{7.52}$$

The RS solution (7.27) is a static solution of these equations, provided that condition (7.28) holds. We now derive linear perturbation equations from Eqs. (7.49)-(7.52) which describe the time evolution of small deviations from RS. To this goal we set

$$N(t,y) = \alpha|y|+n(t,y), \tag{7.53}$$
$$R(t,y) = \alpha|y|+r(t,y), \tag{7.54}$$
$$B(t,y) = b(t,y), \tag{7.55}$$

where $\alpha=-\sqrt{\frac{-\Lambda}{12}}$ and $n(t,y), r(t,y), b(t,y)$ are small at $t=0$. The perturbed metric is

$$\begin{aligned} \mathrm{d}\tilde{s}_5^2 &= \tilde{G}_{MN}\mathrm{d}x^M\mathrm{d}x^N \\ &= -\mathrm{e}^{2\alpha|y|+2n}\mathrm{d}t^2+\mathrm{e}^{2\alpha|y|+2r}\mathrm{d}\vec{x}^2+\mathrm{e}^{2b}\mathrm{d}y^2. \end{aligned} \tag{7.56}$$

We consider an energy-momentum tensor deviating from RS only by a slight mismatch of the brane tension

$$T_{MN}=-\Lambda\tilde{G}_{MN}-\delta(y)\delta^\mu_M\delta^\nu_N\mathrm{e}^{-b}V\tilde{g}_{\mu\nu}, \tag{7.57}$$

with

$$V = \sqrt{-12\Lambda}(1+\Omega), \tag{7.58}$$

where $|\Omega| \ll 1$ parameterizes the perturbation of the brane tension, $\tilde{G}_{MN}$ is the perturbed metric (7.56) and $\tilde{g}_{\mu\nu}$ is its projection onto the brane. Clearly, if already this restricted set of perturbation variables contains an instability, the RS solution is unstable under homogeneous and isotropic perturbations. Inserting this ansatz into equations (7.49)– (7.52) and keeping only first order terms, we find

$$r'' - 4\alpha^2 b + \theta(y)\alpha(4r' - b') - \delta(y)2\alpha(b+\Omega) = 0 \ , \tag{7.59}$$

$$\begin{aligned} &\mathrm{e}^{-2\alpha|y|}(2\ddot{r} + \ddot{b}) - n'' - 2r'' + 12\alpha^2 b - \theta(y)\alpha(4n' + 8r' - 3b') \\ &+\delta(y)6\alpha(b+\Omega) = 0 \ , \end{aligned} \tag{7.60}$$

$$\dot{r}' - \theta(y)\alpha\dot{b} = 0 \ , \tag{7.61}$$

$$\mathrm{e}^{-2\alpha|y|}\ddot{r} + 4\alpha^2 b - \theta(y)\alpha(n' + 3r') = 0 \ , \tag{7.62}$$

where

$$\theta(y) = \begin{cases} +1 & \text{for } y > 0 \\ -1 & \text{for } y < 0 \ . \end{cases} \tag{7.63}$$

The junction conditions are

$$[r'] = [n'] = 2\alpha(b_0 + \Omega) \ . \tag{7.64}$$

Since we want to consider Z_2-symmetric perturbations, we require the functions n, r and b to be symmetric in y. In order to make coordinate-independent statements, we rewrite these equations in a gauge invariant way.

7.4.2 Gauge invariant perturbation equations

Under an infinitesimal coordinate transformation induced by the vector field

$$X = T(t,y)\partial_t + L(t,y)\partial_y, \tag{7.65}$$

the metric perturbations δG_{MN} (corresponding to the first order terms in the metric (7.56)) transform according to

$$\delta G_{MN} \to \delta G_{MN} + \mathcal{L}_X G_{MN}, \tag{7.66}$$

where $\mathcal{L}_X G_{MN}$ is the Lie derivative of the static background metric (7.24). One obtains the following transformation laws for the variables n, r and b:

$$n \to n + \theta(y)\alpha L + \dot{T}, \tag{7.67}$$

$$r \to r + \theta(y)\alpha L, \tag{7.68}$$

$$b \to b + L'. \tag{7.69}$$

Since we require the 04 component of the metric to vanish, it must remain zero under the coordinate transformation. This implies

$$\dot{L} = \mathrm{e}^{2\alpha|y|}T'. \tag{7.70}$$

From Eq. (7.69), together with Z_2 symmetry, one finds that L' must be continuous and symmetric in y. Therefore L must be continuously differentiable and odd in y, which implies $L(t, y = 0) = 0$. Hence, the perturbation r restricted to the brane r_0 is gauge invariant. Note that $L(t, y = 0) = 0$ also follows from Eq. (7.68) and $L \in C^1$. Hence the gauge invariance of r_0 is not a consequence of Z_2 symmetry, but is also preserved for non Z_2-symmetric perturbations. By computing the Lie derivative of the background energy-momentum tensor from Eq. (7.57) one finds that the perturbation of the brane tension Ω is gauge invariant. Condition (7.70) and the symmetry property of L' ensure that there is no energy flow onto or off the brane. With the following set of gauge invariant quantities

$$\Phi \equiv r' - \theta(y)\alpha b \; , \tag{7.71}$$

$$\Psi \equiv n' - \theta(y)\alpha b - \theta(y)\frac{1}{\alpha}\,\mathrm{e}^{-2\alpha|y|}\ddot{r} \; , \tag{7.72}$$

$$r_0 \equiv r(t, y = 0) \; , \tag{7.73}$$

$$\Omega, \tag{7.74}$$

we can rewrite the perturbation equations (7.59)–(7.62) in terms of these variables

$$\Phi' + \theta(y)4\alpha\Phi - \delta(y)2\alpha\Omega = 0 \; , \tag{7.75}$$

$$\Psi' + 2\Phi' + \theta(y)4\alpha(\Psi + 2\Phi) + \delta(y)\left(\frac{2}{\alpha}\ddot{r}_0 - 6\alpha\Omega\right) = 0 \; , \tag{7.76}$$

$$\dot{\Phi} = 0 \; , \tag{7.77}$$

$$\Psi + 3\Phi = 0 \; . \tag{7.78}$$

The junction conditions are

$$[\Phi] = 2\alpha\Omega \; , \qquad\qquad [\Psi] = 2\alpha\Omega - \frac{2}{\alpha}\ddot{r}_0 \; . \tag{7.79}$$

The solutions of equations (7.75) with (7.77) and (7.78) are given by

$$\Phi(y) = -\frac{1}{3}\Psi(y) = \theta(y)\alpha\Omega\mathrm{e}^{-4\alpha|y|} + \Phi_0\mathrm{e}^{-4\alpha|y|} \; . \tag{7.80}$$

Z_2 symmetry requires Φ to be odd in y and thus $\Phi_0 = 0$. Inserting Eq. (7.78) in (7.76) one obtains

$$\ddot{r}_0 = 4\alpha^2\Omega \tag{7.81}$$

and after integration

$$r_0(t) = 2\alpha^2\Omega t^2 + \mathcal{Q}t, \tag{7.82}$$

where $\mathcal{Q}$ is a small but arbitrary integration constant determined by the initial conditions. (An additive constant to r_0 can be absorbed in a redefinition of the spatial coordinates on the brane.) The scale factor on the brane is

$$\mathrm{e}^{2r_0(t)} \simeq 1 + 2r_0(t) = 1 + 4\alpha^2\Omega t^2 + 2\mathcal{Q}t. \tag{7.83}$$

We have thus found a dynamical instability, which is quadratic in time, when the brane tension and the bulk cosmological constant are not fine–tuned. Our statement is valid in every coordinate system as r_0 is gauge invariant. In addition, more surprisingly, in linear perturbation theory there is no constraint on $\mathcal{Q}$, and it cannot be gauged away. This linear instability remains even for $\Omega = 0$, that is if the brane tension is not perturbed at all.

Let us finally discuss our solutions in two particular gauges. As a first gauge condition we set $r' = 0$, which fixes $L' = -\theta(y)\frac{1}{\alpha}r'$. The integration constant on L is determined by the condition $L(t, y = 0) = 0$. (Note that r' contains a θ function and therefore L' is continuous.) For all values of y we have

$$r(t) = 2\alpha^2\Omega t^2 + \mathcal{Q}t. \tag{7.84}$$

Since $b(y) = -\theta(y)\frac{1}{\alpha}\Phi(y)$,

$$b(y) = -\Omega \mathrm{e}^{-4\alpha|y|}. \tag{7.85}$$

From the definition of Ψ it follows that

$$n' = -3\Phi + \theta(y)\alpha b + \theta(y)\frac{1}{\alpha}\,\mathrm{e}^{-2\alpha|y|}\ddot{r}_0\ , \tag{7.86}$$

which can be integrated to give

$$n(t, y) = \Omega \mathrm{e}^{-4\alpha|y|} - 2\Omega \mathrm{e}^{-2\alpha|y|} + \mathcal{N}(t)\ . \tag{7.87}$$

These n, r, and b solve Eqs. (7.59)–(7.62). The integration constant $\mathcal{N}(t)$ can be absorbed in the gauge transformation T. Together with the choice of L', this fixes the gauge, and the solutions are therefore unique up to an additive purely time dependent function to T.

Another possible gauge is $b = 0$. Then, from $r' = \Phi$,

$$r(t, y) = -\frac{\Omega}{4}\,\mathrm{e}^{-4\alpha|y|} + \mathcal{R}(t), \tag{7.88}$$

with

$$\mathcal{R}(t) = r_0(t) + \frac{\Omega}{4} = 2\alpha^2\Omega t^2 + \mathcal{Q}t + \frac{\Omega}{4} \tag{7.89}$$

and

$$n(t, y) = \frac{3}{4}\Omega \mathrm{e}^{-4\alpha|y|} - 2\Omega \mathrm{e}^{-2\alpha|y|} + \mathcal{N}(t)\ . \tag{7.90}$$

Again, the integration constant $\mathcal{N}(t)$ can be gauged away by choosing an appropriate T, and the solutions are uniquely determined by the gauge fixing.

Inserting these solutions in the perturbed metric

$$\mathrm{d}\tilde{s}_5^2 = -\mathrm{e}^{2\alpha|y|+2n}\mathrm{d}t^2 + \mathrm{e}^{2\alpha|y|+2r}\mathrm{d}\vec{x}^2 + \mathrm{e}^{2b}\mathrm{d}y^2, \tag{7.91}$$

we find that the full RS space-time, not only the brane, is unstable against homogeneous perturbations of the brane tension.

We must require the initial perturbations to be small, that is at some initial time $t = 0$, the deviation from RS has to be small for all values of y. In the case of a compact space-time $|y| \leq y_{\max}$, this just requires $|\Omega| \ll \mathrm{e}^{-4\alpha y_{\max}}$ (remember that α is a negative constant). For a noncompact space-time $-\infty < y < \infty$, we have to require $\Omega = 0$. In other words, for $V \not\equiv \sqrt{-12\Lambda}$ there exists no solution which is 'close' to RS in the sense of L^2 or $\sup_y$ at any given initial time.

Finally, we present a geometrical interpretation of the gauge invariant quantities Φ and Ψ. Since the five–dimensional Weyl tensor of the RS solution vanishes, the perturbed Weyl tensor is gauge invariant according to the Steward-Walker lemma [146]. The 0404 component of the Weyl tensor of the perturbed metric (7.56) is up to first order

$$C_{0404} = \frac{1}{2}\,\mathrm{e}^{2\alpha|y|}(n'' - r'' + \theta(y)\alpha(n' - r')) + \frac{1}{2}(\ddot{r} - \ddot{b}), \tag{7.92}$$

which can be expressed in terms of gauge invariant quantities

$$C_{0404} = -\frac{1}{2}\,\mathrm{e}^{\alpha|y|}\left(\mathrm{e}^{\alpha|y|}(\Phi - \Psi)\right)' + \delta(y)\frac{1}{\alpha}\ddot{r}_0. \tag{7.93}$$

All other nonvanishing Weyl components are multiples of C_{0404}:

$$C_{0101} = C_{0202} = C_{0303} = C_{1212} = C_{1313} = C_{2323} = -\frac{1}{3}\,\mathrm{e}^{2\alpha|y|}C_{0404} \tag{7.94}$$

and

$$C_{1414} = C_{2424} = C_{3434} = \frac{1}{3}C_{0404}\ . \tag{7.95}$$

In first order the projected Weyl tensor (defined in Ref. [142]) is $E_{11} = E_{22} = E_{33} = \frac{1}{3}E_{00} = 2\alpha^2\Omega$ with $E^{\mu}{}_{\mu} = 0$. The Weyl–tensor completely vanishes for $\Omega = 0$.

7.5 Results and Conclusions

In this paper we have addressed two main questions: First, we investigated whether the RS fine–tuning condition can be obtained dynamically by some matter component on the brane. As a concrete example, we studied a scalar field on the brane and found that a bulk cosmological constant cannot be cancelled by the potential of the scalar field in a nontrivial way. This result can be generalized for any matter with an equation of state $\omega > -1$.

Second, we studied the stability of the RS model in five dimensions. We have found that the RS solution is unstable under homogeneous and isotropic, but time dependent perturbations. For a small deviation of the fine–tuning condition parameterized by $\Omega \neq 0$, this instability was expected. It is very reminiscent of the instability of the static Einstein universe, where the fluid energy density and the cosmological constant have to satisfy a delicate balance in order to keep the universe static. But even if $\Omega = 0$, there exists a mode ($\mathcal{Q} \neq 0$), which represents an instability in first order perturbation theory. It is interesting to note, that this mode can be absorbed into a motion of the brane, where $\mathcal{Q}$ represents the velocity of the brane. In a cosmological context, our result means that a possible change in the brane tension, *e.g.* during a phase transition, or also quantum corrections to the bulk energy density (see Ref. [61]) could give rise to instabilities of the full five–dimensional space-time. For related work see Refs. [91], [86], [121].

Even if one would consider a dynamical scalar field in the bulk (which does not couple to brane fields), which settles into a vacuum state such that its energy density is constant along the fifth dimension, one would not be able to solve the cosmological constant problem without falling back to some fine–tuning mechanism. From our results we can conclude that in order to have a chance to solve the RS fine tuning problem dynamically, we have to consider fully dynamical *bulk* fields. This can in principle be done with the system of equations, which we have presented in Sec. 7.2, and which also applies to the effective five–dimensional low–energy theory suggested by heterotic M theory [104].

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Chapter 8

On CMB anisotropies in a brane universe

8.1 Introduction

A great deal of our knowledge in cosmology comes from measurements of the cosmic microwave background (CMB), in particular its anisotropies. Recent experiments have provided us with precision data which can be used to constrain the space of cosmological parameters, and thus to confirm or reject cosmological models. From a theoretical point of view, this is possible because perturbation theory in a (3+1)-dimensional universe is very well established.

Of course, one would also like to test brane world models with the CMB, in particular, whether the idea of extra-dimensions is viable at all. Brane world models should leave distinct imprints in the CMB, such that they can hopefully be strongly constrained in the future. Unfortunately, perturbation theory with extra-dimensions is far more complicated than in the standard (3+1)-dimensional case. We have seen in Chap. 6, that the induced Einstein equations (6.37) on the brane are not a closed system. Therefore, we cannot simply perturb those equations, since we miss information on perturbations from the bulk. Instead, we have to perturb the 5-dimensional Einstein equations, which contain the full dynamics of gravity. The brane is then taken into account via Israel's junction conditions.

Recently, a number of works on 5-dimensional perturbation theory have appeared (see [129] and references therein). In these treatments the brane is fixed at some value of the extra-dimensional coordinate, and the bulk is dynamical. When one tries to actually solve the corresponding perturbation equations, it turns out that it is much simpler to consider a static bulk (AdS_5) and a moving brane. Technically, this is due to the fact that if already the background equations are dynamical, the perturbation equations become extremely cumbersome. In this context, one should note that Israel's junction conditions do also apply if the brane is moving.

In this chapter and in the article 'CMB anisotropies from vector perturbations in the bulk' in Chap. 9, we focus on vector perturbations, leaving scalar and tensor modes for future studies. The vector perturbations in the bulk induce vector perturbations on the brane. In the standard cosmology, vector perturbations are known to decay for any initial conditions. We shall see in Chap. 9 that this is completely different for brane worlds, and therefore the study of vector perturbations is very interesting.

The second reason to consider vector perturbations is that massless vector modes in the bulk always remain massless, because they are protected by a gauge symmetry. Therefore, they should be observable at any (in particular low energy) scale. And finally it is fair to say that vector modes are simpler to treat than scalars, since they involve only three instead of seven differential equations.

Compared to the mirage cosmology approach, some progress has been made in that, perturbing Einstein's equations, the back-reaction is now taken into account. Unfortunately, a direct comparison between the two approaches is not possible, as the perturbation on the probe brane was a scalar describing the fluctuations of its embedding.

The outline of this chapter is as follows: in Sec. 8.2 we perturb an AdS_5 bulk and derive the perturbed vector Einstein equations. In the remaining sections, we place ourselves on the brane and recall, how temperature fluctuations in the CMB arise as a result of perturbed photon trajectories (Sec. 8.3), the definition and some properties of the power spectrum (Sec. 8.4), and finally, how the C_ℓ's are actually calculated from the metric and matter perturbations (Sec. 8.5). The treatment in the last three sections is the same as in 4-dimensional standard cosmology.

8.2 Bulk vector perturbations in $4 + 1$ dimensions

8.2.1 Background variables

As mentioned in the introduction, the background is taken to be 5-dimensional anti-de Sitter space-time (AdS_5). We begin by recalling the relevant background tensors. For the AdS_5 metric, we shall use the parametrization given in Eq. (3.63),

$$\begin{aligned} \mathrm{d}s_5^2 &= G_{MN}\mathrm{d}x^M\mathrm{d}x^N \\ &= \frac{r^2}{L^2}(-\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j) + \frac{L^2}{r^2}\mathrm{d}r^2, \end{aligned} \tag{8.1}$$

where r is the coordinate of the extra-dimension, and L is the curvature radius of the AdS_5. The Christoffel symbols are calculated using the formula (1.50),

$$\Gamma^\mu{}_{\nu 4} = \frac{1}{r}\delta^\mu_\nu, \quad \Gamma^4{}_{\mu\nu} = -\frac{r^3}{L^4}\eta_{\mu\nu}, \quad \Gamma^4{}_{44} = -\frac{1}{r}, \tag{8.2}$$

where $\mu = 0, 1, 2, 3$, and the other components are zero. Since AdS_5 is a space-time of constant negative curvature, we can use the expression for the Ricci tensor and the Riemann scalar from Sec. 2.4,

$$R_{MN} = -\frac{4}{L^2}G_{MN}, \quad R = -\frac{20}{L^2}. \tag{8.3}$$

Throughout this and the next chapter, we assume that the bulk is empty, *i.e.* that there is no other form of energy than a (negative) cosmological constant Λ. Then, the 5-dimensional Einstein equations are

$$0 = \mathcal{G}_{MN} + \Lambda G_{MN} = R_{MN} - \frac{1}{2}G_{MN}R + \Lambda G_{MN} = \frac{6}{L^2}G_{MN} + \Lambda G_{MN}, \tag{8.4}$$

thus forcing $\Lambda = -6/L^2$.

8.2.2 Perturbed metric and gauge invariant variables

The splitting of a general 5-dimensional perturbation into scalar, vector, and tensor parts is performed with respect to a 3-dimensional subspace $\mathcal{M}^3$, because then the results are directly comparable to the (3+1)-dimensional cosmology. The scalar, vector, and tensor modes are irreducible components under $SO(3) \times E_3$, which is the group of isometries of the unperturbed space time. Here, $SO(3)$ and E_3 are the groups of spatial rotations and translations, respectively, and the terms 'scalar', 'vector', and 'tensor' correspond to the spin-0, spin-1, and spin-2 representations. The advantage of this decomposition is that, in linear perturbation theory, scalar, vector, and tensor modes evolve independently. In the following, for the reasons mentioned in the introduction, we restrict ourselves to vector modes.

In a 5-dimensional space-time there are three vector variables, B_i, C_i, and E_i needed to parameterize a general vector perturbation,

$$\begin{aligned} \mathrm{d}\tilde{s}_5^2 &= \tilde{G}_{MN}\mathrm{d}x^M\mathrm{d}x^N = (G_{MN} + \delta G_{MN})\mathrm{d}x^M\mathrm{d}x^N \\ &= \frac{r^2}{L^2}\left[-\mathrm{d}t^2 + 2B_i\mathrm{d}t\mathrm{d}x^i + (\delta_{ij} + \nabla_i E_j + \nabla_j E_i)\mathrm{d}x^i\mathrm{d}x^j\right] + 2C_i\mathrm{d}x^i\mathrm{d}r + \frac{L^2}{r^2}\mathrm{d}r^2, \end{aligned} \tag{8.5}$$

where ∇_i denotes the connection in $\mathcal{M}^3$. Since B_i, C_i, and E_i are divergenceless[1] 3-vectors, each of them has only two independent components, and we can set $i = 1, 2$ for a mode with wave vector $\mathbf{k} = (0, 0, k)$. For a more detailed treatment, as well as for the counting of the degrees of freedom, we refer to [129].

Alternatively, we can write Eq. (8.5) in matrix form,

$$(\delta G_{MN}) = \frac{r^2}{L^2}\begin{pmatrix} 0 & B_1 & B_2 & B_3 & 0 \\ B_1 & H_{11} & H_{12} & H_{13} & \frac{L^2}{r^2}C_1 \\ B_2 & H_{12} & H_{22} & H_{23} & \frac{L^2}{r^2}C_2 \\ B_3 & H_{13} & H_{23} & H_{33} & \frac{L^2}{r^2}C_3 \\ 0 & \frac{L^2}{r^2}C_1 & \frac{L^2}{r^2}C_2 & \frac{L^2}{r^2}C_3 & 0 \end{pmatrix}, \tag{8.6}$$

with $H_{ij} \equiv \nabla_i E_j + \nabla_j E_i$.

In order to make coordinate independent statements, the perturbed metric is written in terms of gauge invariant variables. Under an infinitesimal coordinate transformation, induced by the divergenceless vector ε^M,

$$x^M \to x^M + \varepsilon^M, \quad \varepsilon_M = (0, \varepsilon_i, 0), \tag{8.7}$$

with $\varepsilon^i = \delta^{ij}\varepsilon_j$, the metric perturbations transform as

$$\delta G_{MN} \to \delta G_{MN} + \mathcal{L}_\varepsilon G_{MN}, \tag{8.8}$$

where $\mathcal{L}_\varepsilon G_{MN}$ is the Lie derivative of the unperturbed metric in the direction ε^M.

[1]The divergence term yields a scalar.

In terms of the variables B_i, C_i, and E_i, the transformation law (8.8) reads

$$\begin{aligned} B_i &\to B_i + \frac{L^2}{r^2}\partial_t \varepsilon_i, \\ C_i &\to C_i + \partial_r \varepsilon_i - \frac{2}{r}\varepsilon_i, \\ E_i &\to E_i + \frac{L^2}{r^2}\varepsilon_i. \end{aligned} \tag{8.9}$$

Since there are three divergenceless vectors and one gauge freedom (8.7), there exist two divergenceless gauge invariant vectors which are

$$\Sigma_i = B_i - \partial_t E_i, \tag{8.10}$$

$$\Xi_i = C_i - \frac{r^2}{L^2}\partial_r E_i. \tag{8.11}$$

We lose no generality, but considerably simplify the calculations, by setting $E_i = 0$. Then the gauge invariant variables simply reduce to $\Sigma_i = B_i$ and $\Xi_i = C_i$. We shall work with this gauge fixing in the remainder of this section.

8.2.3 Perturbed Einstein equations

The perturbations of the **Christoffel symbols** are

$$\begin{array}{ll} \delta\Gamma^0{}_{00} = 0 & \delta\Gamma^0{}_{0i} = 0 \\ \delta\Gamma^0{}_{04} = 0 & \delta\Gamma^0{}_{ij} = -\frac{1}{2}(\Sigma_{i,j} + \Sigma_{j,i}) \\ \delta\Gamma^0{}_{i4} = \frac{1}{2}\left(\frac{L^2}{r^2}\dot{\Xi}_i - \Sigma_i'\right) & \delta\Gamma^0{}_{44} = 0 \\ \delta\Gamma^k{}_{00} = \dot{\Sigma}_k - \frac{r}{L^2}\Xi_k & \delta\Gamma^k{}_{0i} = \frac{1}{2}\left(\Sigma_{k,i} - \Sigma_{i,k}\right) \\ \delta\Gamma^k{}_{04} = \frac{1}{2}\left(\Sigma_k' + \frac{L^2}{r^2}\dot{\Xi}_k\right) & \delta\Gamma^k{}_{ij} = \frac{r}{L^2}\Xi_k \delta_{ij} \\ \delta\Gamma^k{}_{i4} = \frac{1}{2}\frac{L^2}{r^2}\left(\Xi_{k,i} - \Xi_{i,k}\right) & \delta\Gamma^k{}_{44} = \frac{L^2}{r^2}\left(\Xi_k' + \frac{1}{r}\Xi_k\right) \\ \delta\Gamma^4{}_{00} = 0 & \delta\Gamma^4{}_{0i} = \frac{1}{2}\frac{r^2}{L^2}\left(\dot{\Xi}_i - \frac{r^2}{L^2}\Sigma_i' - \frac{2r}{L^2}\Sigma_i\right) \\ \delta\Gamma^4{}_{04} = 0 & \delta\Gamma^4{}_{ij} = \frac{1}{2}\frac{r^2}{L^2}(\Xi_{i,j} + \Xi_{j,i}) \\ \delta\Gamma^4{}_{i4} = -\frac{r}{L^2}\Xi_i & \delta\Gamma^4{}_{44} = 0. \end{array} \tag{8.12}$$

The perturbations of the **Ricci tensor** are calculated by using its expression in terms of Christoffel symbols (see Eq. (1.56) for the definition),

$$\begin{aligned} \delta R_{MN} = \delta R^A{}_{MAN} = &\delta\Gamma^A{}_{MN,A} + \delta\Gamma^A{}_{BA}\Gamma^B{}_{MN} + \Gamma^A{}_{BA}\delta\Gamma^B{}_{MN} \\ &- \delta\Gamma^A{}_{MA,N} - \delta\Gamma^A{}_{BN}\Gamma^B{}_{MA} - \Gamma^A{}_{BN}\delta\Gamma^B{}_{MA}. \end{aligned} \tag{8.13}$$

This results in

$$
\begin{aligned}
\delta R_{00} &= 0, \\
\delta R_{0i} &= \frac{1}{2}\left(\frac{r^2}{L^2}\dot{\Xi}_i' + 3\frac{r}{L^2}\dot{\Xi}_i - \frac{r^4}{L^4}\Sigma_i'' - \Delta\Sigma_i - 5\frac{r^3}{L^4}\Sigma_i' - 8\frac{r^2}{L^4}\Sigma_i\right), \\
\delta R_{04} &= 0, \\
\delta R_{ij} &= \frac{1}{2}\left(\frac{r^2}{L^2}(\Xi_{i,j}' + \Xi_{j,i}') + 3\frac{r}{L^2}(\Xi_{i,j} + \Xi_{j,i}) - (\dot{\Sigma}_{i,j} + \dot{\Sigma}_{j,i})\right), \\
\delta R_{i4} &= \frac{1}{2}\left(\frac{L^2}{r^2}\ddot{\Xi}_i - \frac{L^2}{r^2}\Delta\Xi_i - \frac{8}{L^2}\Xi_i - \dot{\Sigma}_i'\right), \\
\delta R_{44} &= 0,
\end{aligned}
\tag{8.14}
$$

with the spatial Laplacian

$$
\Delta \equiv \left(\frac{\partial}{\partial x^1}\right)^2 + \left(\frac{\partial}{\partial x^2}\right)^2 + \left(\frac{\partial}{\partial x^3}\right)^2. \tag{8.15}
$$

The **Riemann scalar** is pure scalar and therefore for vector perturbations one has

$$
\delta R = 0. \tag{8.16}
$$

The perturbations of **Einstein's equations** are

$$
\delta R_{MN} - \frac{1}{2}\delta G_{MN} R + \Lambda \delta G_{MN} = 0. \tag{8.17}
$$

Using that the unperturbed Riemann scalar and the 5-dimensional cosmological constant are given by $R = -20/L^2$ and $\Lambda = -6/L^2$ according to Eqs. (8.3) and (8.4), one obtains the following perturbation equations,

$$
\begin{aligned}
&\frac{r^2}{L^2}\dot{\Xi}_i' + 3\frac{r}{L^2}\dot{\Xi}_i - \frac{r^4}{L^4}\Sigma_i'' - \Delta\Sigma_i - 5\frac{r^3}{L^4}\Sigma_i' = 0, \\
&\partial_i\left(\frac{r^2}{L^2}\Xi_j' + 3\frac{r}{L^2}\Xi_j - \dot{\Sigma}_j\right) + \partial_j\left(\frac{r^2}{L^2}\Xi_i' + 3\frac{r}{L^2}\Xi_i - \dot{\Sigma}_i\right) = 0, \\
&\frac{L^2}{r^2}\ddot{\Xi}_i - \frac{L^2}{r^2}\Delta\Xi_i - \dot{\Sigma}_i' = 0,
\end{aligned}
\tag{8.18}
$$

which are the $0i, ij$, and $i4$ components of Einstein's equations. So far, $\Sigma_i = \Sigma_i(t, \mathbf{x}, r)$, and $\Xi_i = \Xi_i(t, \mathbf{x}, r)$. In terms of Fourier components $\Sigma_i = \Sigma_i(t, \mathbf{k}, r)$, and $\Xi_i = \Xi_i(t, \mathbf{k}, r)$, the second equation in (8.18) reads

$$
\frac{r^2}{L^2}\Xi_i' + 3\frac{r}{L^2}\Xi_i - \dot{\Sigma}_i = 0, \tag{8.19}
$$

which is a constraint equation. Its derivative with respect to t can be inserted into the $0i$ equation to eliminate the mixed derivative $\dot{\Xi}_i'$. Similarly, the r derivative is used in $i4$ to eliminate $\dot{\Sigma}_i'$. Then, the first and third equation in (8.18) decouple,

$$\begin{aligned}&\frac{r^4}{L^4}\Sigma_i'' + 5\frac{r^3}{L^4}\Sigma_i' + \Delta\Sigma_i - \ddot{\Sigma}_i = 0,\\&\frac{r^4}{L^4}\Xi_i'' + 5\frac{r^3}{L^4}\Xi_i' + \Delta\Xi_i - \ddot{\Xi}_i + 3\frac{r^2}{L^4}\Xi_i = 0,\end{aligned}\tag{8.20}$$

where Σ_i and Ξ_i are now understood as Fourier modes.

In this derivation we have assumed that the bulk contains no other form of energy than a cosmological constant Λ which is not perturbed. Therefore, any bulk matter perturbations are absent in our equations. The perturbed vector Einstein equations (8.20) can be solved analytically for arbitrary initial conditions. The solutions are given in Eqs. (9.45) and (9.46) in the article 'CMB anisotropies from vector perturbations in the bulk' in Chap. 9.

8.3 Temperature fluctuations in the CMB

8.3.1 Induced vector perturbations on the brane

Our previous derivation of the vector perturbations in the bulk is independent of the presence of a brane. We now consider a 3-brane embedded in the subspace $\mathcal{M}^3$, which is moving along a geodesic of AdS_5 . This setup is the same as in mirage cosmology in chapters 4 and 5, but now the back-reaction will be taken into account: the perturbation equations on the brane are derived via the Israel junction conditions.

From mirage cosmology, we know that the motion of the brane induces a cosmological expansion, where the scale factor is proportional to the radial position of the brane. With a particular embedding, the brane has the properties of a Friedmann-Lemaître universe. For a detailed description of the embedding and the background dynamics we refer to Sec. 9.2.

The vector perturbations in the bulk induce vector perturbations on the brane[2]. Like in the standard cosmology, they are described by two divergenceless variables b_i and e_i,

$$\begin{aligned}\mathrm{d}\tilde{s}_{\mathrm{b}}^2 &= \tilde{g}_{\mu\nu}\mathrm{d}y^\mu\mathrm{d}y^\nu = (g_{\mu\nu} + \delta g_{\mu\nu})\mathrm{d}y^\mu\mathrm{d}y^\nu\\&= -\mathrm{d}\tau^2 + 2ab_i\mathrm{d}\tau\mathrm{d}y^i + a^2(\tau)(\delta_{ij} + \nabla_i e_j + \nabla_j e_i)\mathrm{d}y^i\mathrm{d}y^j,\end{aligned}\tag{8.21}$$

where $y^\mu = (\tau, y^i)$ denote internal coordinates on the brane[3], and τ stands for cosmic time. Since the unperturbed spatial subspace $\mathcal{M}^3$ is flat, its metric is δ_{ij}. The perturbations b_i and e_i are related to B_i, E_i, and C_i by the pull-back of the metric (8.5) onto the brane (see also Eq. (9.65)). For the detailed treatment we

[2]Since the decomposition into scalars, vectors, and tensors was done with respect to $\mathcal{M}^3$, vectors in the bulk correspond to vectors on the brane, and no further scalars arise.

[3]We use y^μ instead of the usual σ^μ, because σ_i will be used to denote the gauge invariant vector perturbation on the brane.

refer to Sec. 9.4; here we just mention that the gauge invariant metric perturbation is

$$\sigma_i = b_i - a\dot{e}_i, \tag{8.22}$$

and the gauge invariant velocity perturbation,

$$\vartheta_i = v_i + a\dot{e}_i. \tag{8.23}$$

The dot denotes a derivative with respect to τ, and v_i is the parametrization of the perturbed 4-velocity of a perfect fluid on the brane, defined in Eq. (9.57). Via the perturbed Israel junction conditions one finds (see also Eq. (9.72) and (9.73))

$$\sigma_i = \sqrt{1+L^2H^2}\Sigma_i + LH\Xi_i, \tag{8.24}$$

$$\sigma_i + \vartheta_i = \frac{\sqrt{1+L^2H^2}}{2L\dot{H}a}\left(a^2\partial_r\Sigma_i - \partial_t\Xi_i\right). \tag{8.25}$$

The gauge invariant quantity $\sigma_i + \vartheta_i$ is known as vorticity. Given the solutions for Σ_i and Ξ_i, these two quantities are the input for the calculation of the vector CMB anisotropies in a brane universe. Once σ_i and ϑ_i are known in terms of brane coordinates (τ, y^i), one can forget about their 5-dimensional origin and apply the standard theory of CMB anisotropies in $3+1$ dimensions. This is what we are going to do in the remainder of this chapter.

8.3.2 Sources of CMB anisotropies

Photons incident from different directions in the sky have nearly the same temperature of 2.73 K. This almost uniform background radiation is the strongest indication that our universe was isotropic already at the time of decoupling of photons and baryons. Today, various experiments are able to measure the temperature fluctuations in the CMB with great accuracy. The major effects and sources giving rise to those CMB anisotropies are

1. **Density fluctuations** of photons at the time η_E of emission from the last scattering surface. A denser region emits hotter photons according to the Stefan–Bolzmann law $\rho_\gamma \sim T^4$.

2. Fluctuations in the gravitational potential. Photons emitted in a potential well get red-shifted while they climb out of it. Thus, today they are slightly colder. This is called the **Sachs–Wolfe effect**.

3. Since the moment of decoupling, the photons have travelled through the perturbed geometry (8.21). All the gains and losses in energy along a trajectory from η_E to today, η_R, have to be summed up. This is called the **integrated Sachs–Wolfe effect**.

4. A **Doppler shift**, because the matter that last interacted with the CMB photons was in relative motion to an comoving observer today.

Further sources of CMB anisotropies can be found *e.g.* in Ref. [154]. The first two effects are associated with perturbations of scalar quantities such as the photon density contrast. Therefore, we do not take them into account here. The last two effects involve also vector perturbations, and so they are important for our calculation of the CMB anisotropies in brane worlds later on. One has to keep in mind that in our treatment the vector perturbations are created in the bulk. When projected onto the brane, they manifest themselves in the same way as the sources listed above. We neglect any vector perturbation that have their origin on the brane itself.

The power spectrum that we are going to calculate in the article in Chap. 9 looks very different from the standard one, as physics in the extra-dimension leave a distinct imprint in the CMB. We can use this to strongly constrain at least a certain class of brane world models with the CMB.

In the following two paragraphs, we investigate in detail the temperature fluctuations associated with the integrated Sachs–Wolfe effect.

8.3.3 Temperature fluctuations as a function of the perturbation variables

Consider a 4-dimensional Lorentzian manifold $\mathcal{M}^4$ with internal coordinates $y^\mu = (\eta, \mathbf{y})$, η denoting conformal time, and a perturbed metric $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We want to investigate the trajectory of a photon through $\mathcal{M}^4$, starting at emission on the last scattering surface and ending in a telescope today. In parametric form, this trajectory can be written as $y^\mu(\lambda) = (\eta(\lambda), \mathbf{y}(\lambda))$, where λ denotes an affine parameter. The perturbed 4-momentum of the photon is

$$\tilde{k}^\mu = \frac{\mathrm{d}y^\mu}{\mathrm{d}\lambda}. \tag{8.26}$$

The unperturbed 4-momentum is parameterized as

$$k^\mu = (\nu, -\nu\mathbf{n}), \tag{8.27}$$

where $\mathbf{n} = (n^1, n^2, n^3)$ is the direction of observation. Since k^μ is a light-like 4-vector ($\eta_{\mu\nu}k^\mu k^\nu = 0$), $\mathbf{n}$ must be normalized: $\mathbf{n}^2 = 1$. The perturbed photon momentum is parameterized as

$$\tilde{k}^\mu = k^\mu + \delta k^\mu = (\nu, -\nu\mathbf{n}) + (\delta k^0, \delta\mathbf{k}). \tag{8.28}$$

We shall see later on, that δk^0 has a vector component arising from the metric perturbation σ_i. We also need to specify the 4-velocity of an observer comoving with the baryon fluid,

$$\tilde{u}^\mu = u^\mu + \delta u^\mu = \left(\frac{1}{a}, \mathbf{0}\right) + \left(0, \frac{v_i}{a}\right), \tag{8.29}$$

$$\tilde{u}_\mu = u_\mu + \delta u_\mu = (-a, \mathbf{0}) + (0, a(v_i + b_i)). \tag{8.30}$$

The scale factor a shows up in these expressions, since we have used conformal time[4]. Furthermore, we define $T_R = T(\eta_R, \mathbf{n})$ to be the temperature of photons observed today in the direction $\mathbf{n}$, and $T_E = T(\eta_E, \mathbf{x}_E)$ to be the temperature at the emission point $(\eta_E, \mathbf{x}_E)$ on the last scattering surface. The ratio of the two temperatures is given by

$$\frac{T_R}{T_E} = \frac{(\tilde{k}^\mu \tilde{u}_\mu)_R}{(\tilde{k}^\mu \tilde{u}_\mu)_E}. \tag{8.31}$$

From Eqs. (8.28) and (8.30), one finds

$$\tilde{k}^\mu \tilde{u}_\mu = -a\nu \left(1 + n^i(v_i + b_i) + \frac{1}{\nu}\delta k^0\right). \tag{8.32}$$

The combination $v_i + b_i$ is equal to $\vartheta_i + \sigma_i$ according to Eqs. (8.22) and (8.23) and therefore gauge invariant. The denominator in Eq. (8.31) is Taylor expanded in the perturbed quantities such that Eq. (8.31) becomes, up to first order,

$$\frac{T_R}{T_E} = \frac{a_R}{a_E}\frac{\nu_R}{\nu_E}\left(1 + n^i(\vartheta_i + \sigma_i)\Big|_E^R + \frac{1}{\nu}\delta k^0\Big|_E^R\right). \tag{8.33}$$

The $T \sim 1/a$ form of the unperturbed temperature curve is recovered by[5] setting $\nu \sim 1/a^2$. Since the last scattering surface is not a surface of constant time, but a surface of constant free electron density, we make the following expansions around the mean time of emission, $\bar{\eta}_E$,

$$\begin{aligned}
&\eta_E = \bar{\eta}_E + \delta\eta_E,\\
&a_E \equiv a(\eta_E) = a(\bar{\eta}_E) + a'(\bar{\eta}_E)\delta\eta_E \equiv a(\bar{\eta}_E)\left(1 + \mathcal{H}(\bar{\eta}_E)\right)\delta\eta_E,\\
&T_E \equiv T(\eta_E, \mathbf{x}_E) = T(\bar{\eta}_E) + T'(\bar{\eta}_E, \mathbf{x}_E)\delta\eta_E \equiv T(\bar{\eta}_E) + \delta T_E(\eta_E, \mathbf{x}_E),\\
&T_R \equiv T(\eta_R, \mathbf{n}) = T(\eta_R) + T'(\eta_R, \mathbf{x}_R)\delta\eta_R \equiv T(\eta_R) + \delta T_R(\eta_R, \mathbf{n}).
\end{aligned} \tag{8.34}$$

Here, the prime denotes a derivative with respect to conformal time η. The temperature $T(\bar{\eta}_E) \equiv \bar{T}_E$ is a spatial average over the last scattering surface in the sense of an ensemble average, whereas $T(\eta_R) \equiv \bar{T}_R$ represents an average over all directions of observations in a unique measurement. Under the assumption of ergodicity, the two ways of averaging are equivalent. From (8.34) one derives

$$\frac{T_R}{T_E} = \frac{\bar{T}_R}{\bar{T}_E}\left(1 + \frac{\delta T_R}{\bar{T}_R} - \frac{\delta T_E}{\bar{T}_E}\right). \tag{8.35}$$

The left hand side of Eq. (8.33) is now replaced by Eq. (8.35). Isolating $\delta T_R/\bar{T}_R$ in the resulting expression and using

$$\frac{a(\bar{\eta}_E)}{a_R} = \frac{\bar{T}_R}{\bar{T}_E} \tag{8.36}$$

[4]As a physical quantity, the four-velocity is defined as $u^\mu = \frac{\mathrm{d}y^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}y^\mu}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = \frac{\mathrm{d}y^\mu}{\mathrm{d}\eta}\frac{1}{a}$.

[5]corresponding to the affine parametrization $\frac{\mathrm{d}\eta}{\mathrm{d}\lambda} \sim \frac{1}{a^2}$.

leads to

$$\frac{\delta T_{R}}{\bar{T}_{R}} = n^{i}(\vartheta_i + \sigma_i)\Big|_{E}^{R} + \frac{1}{\nu}\delta k^0\Big|_{E}^{R} + \frac{\delta T_{E}}{\bar{T}_{E}} + \mathcal{H}(\bar{\eta}_{E})\delta\eta_{E}. \tag{8.37}$$

The left hand side depends only on the direction of observation $\mathbf{n}$ and the time of observation η_{R}, the right hand side depends on the point of emission and the geometry along which the photon travels to us. The term $\delta T_{E}/\bar{T}_{E}$ in Eq. (8.37) has to be evaluated at $(\eta_{E}, \mathbf{x}_{E})$. Using the definition of δT_{E} in Eqs. (8.34) and the relation (8.36) one finds

$$\frac{\delta T_{E}}{\bar{T}_{E}}(\eta_{E}, \mathbf{x}_{E}) = \frac{\delta T_{E}}{\bar{T}_{E}}(\bar{\eta}_{E}, \mathbf{x}_{E}) - \mathcal{H}(\bar{\eta}_{E})\delta\eta_{E}, \tag{8.38}$$

such that in Eq. (8.37) the term in $\mathcal{H}$ is cancelled. The remaining term can be written as

$$\frac{\delta T_{E}}{\bar{T}_{E}}(\bar{\eta}_{E}, \mathbf{x}_{E}) = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} \tag{8.39}$$

according to the Stephan–Boltzmann law $\rho_\gamma \sim T^4$. This term is a scalar and therefore we do not consider it in our treatment. With these steps Eq. (8.37) becomes

$$\frac{\delta T_{R}}{\bar{T}_{R}} = n^{i}(\vartheta_i + \sigma_i)\Big|_{E}^{R} + \frac{1}{\nu}\delta k^0\Big|_{E}^{R}. \tag{8.40}$$

8.3.4 The perturbed geodesic equation

We are left with the calculation of the term $\delta k^0/\nu$. When a photon travels through the perturbed geometry on $\mathcal{M}^4$, its 4-momentum is continuously deflected. Instantaneously, this is measured by the quantity δk^μ, which is found by solving the perturbed geodesic equation. The calculation can be considerably simplified by noting that two conformally equivalent metrics, $\mathrm{d}s^2 = a^2\mathrm{d}\hat{s}^2$, have the same light-like geodesics. We therefore need only consider the perturbed metric,

$$\mathrm{d}\hat{s}^2 = (\eta_{\mu\nu} + \hat{h}_{\mu\nu})\mathrm{d}y^\mu\mathrm{d}y^\nu = -\mathrm{d}\eta^2 + 2b_i\mathrm{d}\eta\mathrm{d}y^i + (\delta_{ij} + \nabla_i e_j + \nabla_j e_i)\mathrm{d}y^i\mathrm{d}y^j. \tag{8.41}$$

The perturbed 'normalized' 4-momentum of the photon is then

$$\hat{\tilde{k}}^\mu = \frac{\mathrm{d}y^\mu}{\mathrm{d}\hat{\lambda}}, \tag{8.42}$$

where $\hat{\lambda}$ denotes an affine parameter, and the hat indicates that a quantity is associated with the metric (8.41). We shall use the parameterization

$$\hat{\tilde{k}}^\mu = \hat{k}^\mu + \delta\hat{k}^\mu = (1, \hat{k}^i) + (\delta\hat{k}^0, \delta\hat{k}^i), \tag{8.43}$$

with the light-like normalization $(\hat{k}^i)^2 = 1$.

The perturbed geodesic equation reads

$$\frac{\mathrm{d}\hat{\tilde{k}}^\mu}{\mathrm{d}\hat{\lambda}} + \hat{\tilde{\Gamma}}^\mu{}_{\alpha\beta}\hat{\tilde{k}}^\alpha\hat{\tilde{k}}^\beta = 0. \tag{8.44}$$

Since the background is time-independent, the unperturbed Christoffel symbols vanish, and the first order perturbation of the geodesic equation is simply

$$\frac{\mathrm{d}}{\mathrm{d}\hat{\lambda}}\delta\hat{k}^{\mu} + \delta\hat{\Gamma}^{\mu}_{\alpha\beta}\hat{k}^{\alpha}\hat{k}^{\beta} = 0, \tag{8.45}$$

with

$$\delta\hat{\Gamma}^{\mu}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}(\hat{h}_{\nu\alpha,\beta} + \hat{h}_{\nu\beta,\alpha} - \hat{h}_{\alpha\beta,\nu}). \tag{8.46}$$

Using that $\hat{h}_{\nu\alpha,\beta}\hat{k}^{\beta} = \frac{\mathrm{d}}{\mathrm{d}\hat{\lambda}}\hat{h}_{\nu\alpha}$ and that, from the unperturbed geodesic equation, $\frac{\mathrm{d}\hat{k}^{\mu}}{\mathrm{d}\hat{\lambda}} = 0$, one can integrate Eq. (8.45) from the moment of emission to the moment of reception. This results in

$$\delta\hat{k}^{\mu}\Big|_{R}^{E} = -\eta^{\mu\nu}\hat{h}_{\nu\alpha}\hat{k}^{\alpha}\Big|_{R}^{E} + \frac{1}{2}\eta^{\mu\nu}\int_{R}^{E}\mathrm{d}\hat{\lambda}\,\hat{h}_{\alpha\beta,\nu}\hat{k}^{\alpha}\hat{k}^{\beta}. \tag{8.47}$$

Without loss of generality, we may now choose $\hat{h}_{ij} \equiv \nabla_i e_j + \nabla_j e_i = 0$. Then $\hat{h}_{0i} \equiv b_i = \sigma_i$. Furthermore, $\hat{h}_{00} = 0$ as we are considering only vectors. Upon inserting this into Eq. (8.47), one finds for the zero component

$$\delta\hat{k}^{0}\Big|_{R}^{E} = \sigma_i\hat{k}^{i}\Big|_{R}^{E} - \int_{R}^{E}\mathrm{d}\hat{\lambda}\,(\partial_\eta\sigma_i)\hat{k}^{i}. \tag{8.48}$$

Comparing with Eq. (8.27), we make the identifications $\hat{k}^{\mu} = (1, -n^i)$ and $\hat{\lambda} = \eta$. Then $\delta\hat{k}^{0} = \delta k^{0}/\nu$. Inserting the result (8.48) into (8.40), one finds the final result

$$\frac{\delta T_{R}}{\bar{T}_{R}} = n^i\vartheta_i\big|_{E}^{R} + \int_{E}^{R}\mathrm{d}\eta\,(\partial_\eta\sigma_i)n^i. \tag{8.49}$$

The first term on the right hand side is the perturbation in the Doppler shift due to the relative motion of the observer and the emitter. The second term is the integrated Sachs–Wolfe (ISW) effect: the changes of the geometry perturbation along the trajectory, $\partial_\eta\sigma_i$, are projected onto the 'line of sight' n^i, and the resulting corrections are summed up.

Sometimes it is useful to write the result (8.49) in an alternative form. To that end the total derivative along the trajectory is written as

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\sigma_i(\eta, \mathbf{y}(\eta)) = \partial_\eta\sigma_i + \frac{\partial\sigma_i}{\partial y^j}\frac{\partial y^j}{\partial\eta} = \partial_\eta\sigma_i - (\partial_j\sigma_i)n^j. \tag{8.50}$$

Upon partial integration, Eq. (8.49) becomes

$$\frac{\delta T_{R}}{\bar{T}_{R}} = n^i(\vartheta_i + \sigma_i)\big|_{E}^{R} + \int_{E}^{R}\mathrm{d}\eta\,(\partial_j\sigma_i)n^j n^i. \tag{8.51}$$

This form makes the vorticity appear explicitly.

Eqs. (8.49) and (8.51) are the final results for the temperature fluctuations in the CMB in the direction $\mathbf{n}$ due to vector perturbations in the standard universe or in a brane world. Notice, that $\delta T_{R}/\bar{T}_{R}$ is written entirely in terms of gauge invariant variables, as it must be the case for a measurable quantity.

8.4 Observation of the temperature fluctuations

In practice, measurements of the temperature fluctuations in the CMB are expressed in terms of the angular power spectrum C_ℓ, where ℓ is the multipole number. In this section, we explore the link between $\delta T_R/\bar{T}_R$ and C_ℓ, following Refs. [154], [152], and [53]. Our mathematical conventions are those of Ref. [70]. From the angular power spectrum a variety of cosmological parameters can be extracted.

The temperature fluctuations can be expanded into spherical harmonics Y_ℓ^m, which form a complete set on the 2-sphere:

$$\frac{\delta T_R}{\bar{T}_R} \equiv \frac{\delta T}{T}(\mathbf{n}, \eta_R, \mathbf{x}_R) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} a_{\ell m}(\eta_R, \mathbf{x}_R) Y_\ell^m(\mathbf{n}), \tag{8.52}$$

where the coefficients $a_{\ell m}$ are given by

$$a_{\ell m}(\eta_R, \mathbf{x}_R) = \int_{S^2} \mathrm{d}\Omega \frac{\delta T}{T}(\mathbf{n}, \eta_R, \mathbf{x}_R) Y_\ell^m(\mathbf{n})^*. \tag{8.53}$$

Here, $\bar{T}_R \equiv T = 2.728 \pm 0.002\,\mathrm{K}$ denotes the mean temperature of the CMB. In the following, we are omitting the arguments η_R and $\mathbf{x}_R$, which indicate the time of observation and the position of the observer.

The degree to which the temperature of photons incident from $-\mathbf{n}$ and $-\mathbf{n}'$ is correlated, is measured by the two point function

$$\left\langle \frac{\delta T}{T}(\mathbf{n}) \frac{\delta T}{T}(\mathbf{n}') \right\rangle. \tag{8.54}$$

The symbol $\langle\rangle$ denotes the hypothetical average over an ensemble of universes, *i.e.* the sum over of all possible different realizations of the CMB for constant $\cos\theta = \mathbf{n} \cdot \mathbf{n}'$. From the definition (8.52) one finds

$$\left\langle \frac{\delta T}{T}(\mathbf{n}) \frac{\delta T}{T}(\mathbf{n}') \right\rangle = \sum_{\ell,\ell'} \sum_{m,m'} \langle a_{\ell m} a_{\ell' m'} \rangle\, Y_\ell^m(\mathbf{n}) Y_\ell^m(\mathbf{n}')^*, \tag{8.55}$$

where,

$$\langle a_{\ell m} a_{\ell' m'} \rangle = \int \mathrm{d}\Omega \int \mathrm{d}\Omega' \left\langle \frac{\delta T}{T}(\mathbf{n}) \frac{\delta T}{T}(\mathbf{n}') \right\rangle Y_\ell^m(\mathbf{n})^* Y_{\ell'}^{m'}(\mathbf{n}'). \tag{8.56}$$

The coefficients $a_{\ell m}$ originate from a stochastic process, *e.g.* quantum fluctuations or phase transitions, that generates perturbations in the early universe. Assuming that this process is statistically homogeneous and isotropic, one can write

$$\langle a_{\ell m} a_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell, \tag{8.57}$$

where the quantity C_ℓ does not depend on $\mathbf{x}_R$ (homogeneity) or m (isotropy). The C_ℓ's are the CMB power spectrum. Using Eq. (8.57), we obtain

$$\left\langle \frac{\delta T}{T}(\mathbf{n})\frac{\delta T}{T}(\mathbf{n}')\right\rangle = \sum_\ell C_\ell \underbrace{\sum_m Y_\ell^m(\mathbf{n})Y_\ell^m(\mathbf{n}')^*}_{=\frac{2\ell+1}{4\pi}P_\ell(\mathbf{n}\cdot\mathbf{n}')} = \frac{1}{4\pi}\sum_\ell (2\ell+1)C_\ell P_\ell(\mathbf{n}\cdot\mathbf{n}'), \tag{8.58}$$

where we have used the addition theorem for spherical harmoincs, and the P_ℓ's are Legendre's polynomials. In a spatially flat universe, the index ℓ is related to the angle θ between two points on the sky in the directions $\mathbf{n}$ and $\mathbf{n}'$ by[6]

$$\theta = \frac{180^\circ}{\ell}. \tag{8.59}$$

Thus the quantity $(2\ell+1)C_\ell$ appearing in Eq. (8.58) is the amplitude of the temperature correlation on the angular scale θ. The CMB power spectrum has its highest peak at $\ell \simeq 220$, which means that the strongest temperature correlation is between points at an angular distance $\theta = \frac{180^\circ}{220} \simeq 0.8^\circ$.

The relation (8.58) can be inverted to give C_ℓ as a function of the temperature correlator. Multiplying both sides with $P_{\ell'}(\mu)$ (where $\mu = \mathbf{n}\cdot\mathbf{n}'$), integrating over the interval $[-1,1]$, and using the orthogonality relation $\int_{-1}^1 \mathrm{d}\mu P_\ell(\mu)P_{\ell'}(\mu) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$, one finds

$$C_\ell = 2\pi \int_{-1}^1 \mathrm{d}\mu \left\langle \frac{\delta T}{T}(\mathbf{n})\frac{\delta T}{T}(\mathbf{n}')\right\rangle P_\ell(\mu). \tag{8.60}$$

In this 'hypothetical' treatment $\delta T/T, a_{\ell m}$, and C_ℓ are random variables in an ensemble of universes. Since in practice we cannot reproduce our universe many times, one makes the ergodic hypothesis, that the ensemble average can be replaced by a spatial average. For a given multipole ℓ, there are $2\ell+1$ independent numbers $a_{\ell m}$ over which this average can be taken. One thus defines an estimator for the observed CMB power spectrum by

$$C_\ell^{\rm obs} \equiv \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell} |a_{\ell m}^{\rm obs}|^2. \tag{8.61}$$

The $a_{\ell m}^{\rm obs}$ are assumed to be gaussian random variables following a χ^2 distribution with $2\ell+1$ degrees of freedom. Then the expectation value of $C_\ell^{\rm obs}$ is equal to the hypothetical value C_ℓ, and the variance is

$$\frac{\sqrt{(C_\ell^{\rm obs})^2 - C_\ell^2}}{C_\ell} = \sqrt{\frac{2}{2\ell+1}}. \tag{8.62}$$

[6]This conversion is similar to that between real space and Fourier space quantities, *e.g.* $\lambda = 2\pi/k$.

For large ℓ, the estimator defined in (8.61) is good, whereas for small ℓ (large scales) the variance is a fundamental limitation on the precision of the measurements. This is called 'cosmic variance' and is due to the fact that we can observe only a single realization of the CMB and this only from one given position.

Since in linear perturbation theory scalar, vector, and tensor perturbations are independent, also the corresponding $a_{\ell m}$'s and C_ℓ's are independent

$$\langle a^{S}_{\ell m} a^{V}_{\ell' m'}\rangle = \langle a^{S}_{\ell m} a^{T}_{\ell' m'}\rangle = \langle a^{V}_{\ell m} a^{T}_{\ell' m'}\rangle = 0. \tag{8.63}$$

8.5 The C_ℓ's of vector perturbations

With the definitions from the previous section, we can now calculate the angular power spectrum C_ℓ for vector perturbations ϑ_i and σ_i. The basic idea is to insert expression (8.49) for $\delta T/T$ into the correlator $\langle a_{\ell m} a_{\ell' m'}\rangle$ given in Eq. (8.56), and carry out the integrals over the sphere. In detail, the procedure is as follows. First, $\delta T/T$ is expanded into vector Fourier modes,

$$\begin{aligned}
\vartheta_i(\eta, \mathbf{x}) &= \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 k\, \hat{\vartheta}_i(\eta, \mathbf{k}) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \\
\sigma_i'(\eta, \mathbf{x}) &= \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 k\, \hat{\sigma}_i'(\eta, \mathbf{k}) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}},
\end{aligned} \tag{8.64}$$

which allows us to treat each $\mathbf{k}$ independently. For a fixed $\mathbf{k}$, one defines a set of orthogonal basis vectors $(\mathbf{k}, \mathbf{e}^+, \mathbf{e}^\times)$ as shown in Fig. 8.1 by

$$\mathbf{e}^\lambda \cdot \mathbf{k} = 0, \quad \mathbf{e}^\lambda \cdot \mathbf{e}^{\lambda'} = \delta_{\lambda\lambda'}, \quad \lambda, \lambda' = +, \times. \tag{8.65}$$

Since a photon is transversely polarized, the amplitudes $\hat{\vartheta}_i(\eta, \mathbf{k})$ and $\hat{\sigma}_i'(\eta, \mathbf{k})$ can be decomposed into

$$\begin{aligned}
\hat{\vartheta}_i(\eta, \mathbf{k}) &= \sum_{\lambda=+,\times} \hat{\vartheta}_\lambda(\eta, k) a_\lambda(\mathbf{k}) e_i^\lambda(\mathbf{k}), \\
\hat{\sigma}_i'(\eta, \mathbf{k}) &= \sum_{\lambda=+,\times} \hat{\sigma}_\lambda'(\eta, k) a_\lambda(\mathbf{k}) e_i^\lambda(\mathbf{k}),
\end{aligned} \tag{8.66}$$

where $a_\lambda(\mathbf{k})$ are random amplitudes for each mode $\mathbf{k}$, and e_i^λ are the components of $\mathbf{e}^\lambda$.

We start our calculation from Eq. (8.49),

$$\frac{\delta T}{T}(\mathbf{n}) = -n^i \vartheta_i(\eta_E, \mathbf{x}_E) + \int_E^R \mathrm{d}\eta\, (\partial_\eta \sigma_i) n^i. \tag{8.67}$$

The term $\vartheta_i(\eta_R, \mathbf{x}_R)$, which is a dipole corresponding to the observers (our) motion, has been omitted since it can be set to zero by a redefinition of the coordi-

Figure 8.1: The direction of observation, $\mathbf{n}$, and the set of orthogonal basis vectors $(\mathbf{k}, \mathbf{e}^{+}, \mathbf{e}^{\times})$ over which the amplitudes $\hat{\vartheta}_i(\eta, \mathbf{k})$ and $\hat{\sigma}'_i(\eta, \mathbf{k})$ are decomposed.

nates. On inserting the above Fourier decompositions one finds for the temperature correlator,

$$\begin{aligned}\left\langle \frac{\delta T}{T}(\mathbf{n})\frac{\delta T}{T}(\mathbf{n}')\right\rangle &= \frac{1}{(2\pi)^3}\int \mathrm{d}^3k \int \mathrm{d}^3k' \sum_{\lambda,\lambda'} \Big\{ - n^i e_i^{\lambda} \hat{\vartheta}_{\lambda} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}_E} \\ &+ \int_E^R \mathrm{d}\eta\, n^i e_i^{\lambda} \hat{\sigma}'_{\lambda} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \Big\} \times \{\mathbf{k}\to\mathbf{k}', \mathbf{n}\to\mathbf{n}', \lambda\to\lambda'\}^* \left\langle a_{\lambda}(\mathbf{k}) a_{\lambda'}(\mathbf{k}')\right\rangle .\end{aligned} \tag{8.68}$$

Assuming that the process generating the perturbations was isotropic, the correlator of the $a_\lambda(\mathbf{k})$ satisfies

$$\langle a_\lambda(\mathbf{k}) a_{\lambda'}(\mathbf{k}')\rangle = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}. \tag{8.69}$$

Next, the temperature correlator (8.68) has to be integrated over the 2-sphere. Therefore, it is useful to write $n^i e_i^\lambda$ and $\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}$ in terms of angular variables. With respect to the basis $(\mathbf{k}, \mathbf{e}^{+}, \mathbf{e}^{\times})$, one has $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where we use θ for the angle between $\mathbf{k}$ and $\mathbf{n}$ (rather than between $\mathbf{n}$ and $\mathbf{n}'$ as before). Setting $\mathbf{e}^{+} = (1,0,0), \mathbf{e}^{\times} = (0,1,0)$, and $\mathbf{k} = (0,0,k)$ one finds

$$\begin{aligned} n^i e_i^\lambda &= \delta_\times^\lambda \sin\theta\sin\phi + \delta_+^\lambda \sin\theta\cos\phi \\ &= \delta_\times^\lambda \sin\theta \frac{1}{2i}\left(\mathrm{e}^{i\phi} - \mathrm{e}^{-i\phi}\right) + \delta_+^\lambda \sin\theta\frac{1}{2}\left(\mathrm{e}^{i\phi} + \mathrm{e}^{-i\phi}\right). \end{aligned} \tag{8.70}$$

From the definition $\frac{\mathrm{d}x^\mu}{\mathrm{d}\eta} = \hat{k}^\mu = (1, -n^i)$ in the previous section, we have $\mathbf{x} =$

$\mathbf{n}(\eta_R - \eta)$, and hence

$$\mathbf{k} \cdot \mathbf{x} = \mathbf{k} \cdot \mathbf{n}(\eta_R - \eta) = k(\eta_R - \eta)\cos\theta. \tag{8.71}$$

Eqs. (8.69), (8.70), and (8.71) are inserted into Eq. (8.68), and the k'-integral and the sum over λ' drop out.

When the temperature correlator is integrated over the sphere (see Eq. (8.56)), the quantity

$$\mathcal{I}^{\pm}_{\ell m}(\eta, k) \equiv \int \mathrm{d}\Omega\, Y_\ell^m(\mathbf{n})^* \sin\theta \mathrm{e}^{\pm i\phi} \mathrm{e}^{ik(\eta_R-\eta)\cos\theta} \tag{8.72}$$

appears. The plane wave $\mathrm{e}^{ik(\eta_R-\eta)\cos\theta}$ is decomposed into spherical Bessel functions and Legendre polynomials,

$$\begin{aligned}\mathrm{e}^{ik(\eta_R-\eta)\cos\theta} &= \sum_{\ell'=0}^{\infty} i^{\ell'}(2\ell'+1) j_{\ell'}[k(\eta_R-\eta)] P_{\ell'}(\cos\theta)\\ &= \sum_{\ell'=0}^{\infty} i^{\ell'}(2\ell'+1) j_{\ell'}[k(\eta_R-\eta)] \sqrt{\frac{4\pi}{2\ell'+1}} Y^0_{\ell'}(\mathbf{n}),\end{aligned} \tag{8.73}$$

such that $\mathcal{I}^{\pm}_{\ell m}$ becomes

$$\mathcal{I}^{\pm}_{\ell m} = \sum_{\ell'} i^{\ell'} \sqrt{4\pi(2\ell'+1)} j_{\ell'}[k(\eta_R-\eta)] \int \mathrm{d}\Omega\, Y_\ell^m(\mathbf{n})^* Y^0_{\ell'}(\mathbf{n}) \sin\theta \mathrm{e}^{\pm i\phi}. \tag{8.74}$$

To solve the integral on the right, one uses the recursion relations for spherical harmonics to write

$$\sin\theta \mathrm{e}^{\pm i\phi} Y^0_{\ell'} = \mp \sqrt{\frac{(\ell'+2)(\ell'+1)}{(2\ell'+3)(2\ell'+1)}} Y^{\pm 1}_{\ell'+1} \pm \sqrt{\frac{\ell'(\ell'-1)}{(2\ell'+1)(2\ell'-1)}} Y^{\pm 1}_{\ell'-1}. \tag{8.75}$$

The remaining integral turns out to be simply the orthogonality relation

$$\int \mathrm{d}\Omega\, Y_\ell^m(\mathbf{n})^* Y^1_{\ell'\pm 1}(\mathbf{n}) = \delta_{\ell,\ell'\pm 1}\delta_{m,1}. \tag{8.76}$$

Putting everything together, one finds

$$\mathcal{I}^{\pm}_{\ell m} = \pm\delta_{m,\pm 1}\, i^{\ell+1} \sqrt{\frac{4\pi}{2\ell+1}} \sqrt{\ell(\ell+1)} \left\{ j_{\ell+1}[k(\eta_R-\eta)] + j_{\ell-1}[k(\eta_R-\eta)] \right\}, \tag{8.77}$$

which can be simplified using the relation

$$j_{\ell+1}(x) + j_{\ell-1}(x) = \frac{2\ell+1}{x} j_\ell(x) \tag{8.78}$$

to give

$$\mathcal{I}^{\pm}_{\ell m} = \pm\delta_{m,\pm 1}\, i^{\ell+1} \sqrt{4\pi(2\ell+1)} \sqrt{\ell(\ell+1)} \frac{j_\ell[k(\eta_R-\eta)]}{k(\eta_R-\eta)}. \tag{8.79}$$

After this side-calculation we now resume the calculation of the the power spectrum C_ℓ. The temperature correlator (8.68) is inserted into Eq. (8.56) for $\ell = \ell'$ and $m = m'$. Since all information about the stochastic correlations are contained in $a_\lambda(\mathbf{k})$, and because of the assumption (8.69), we can replace $\langle a_{\ell m} a_{\ell m} \rangle$ by $|a_{\ell m}|^2$. Then Eq. (8.56) becomes

$$\langle |a_{\ell m}|^2 \rangle = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k \frac{1}{4} \Big| i\hat{\vartheta}_\times \left(\mathcal{I}^+_{\ell m} - \mathcal{I}^-_{\ell m}\right) - \hat{\vartheta}_+ \left(\mathcal{I}^+_{\ell m} + \mathcal{I}^-_{\ell m}\right)$$
$$+ \int_E^R \mathrm{d}\eta \left[-i\hat{\sigma}'_\times \left(\mathcal{I}^+_{\ell m} - \mathcal{I}^-_{\ell m}\right) + \hat{\sigma}'_+ \left(\mathcal{I}^+_{\ell m} + \mathcal{I}^-_{\ell m}\right)\right] \Big|^2 . \tag{8.80}$$

We have carried out the remaining sum over λ to remove $\delta^\lambda_+, \delta^\lambda_\times$ coming from Eq. (8.70). The amplitudes $\hat{\vartheta}_\times, \hat{\vartheta}_+, \hat{\sigma}'_\times, \hat{\sigma}'_+$ were defined in Eq. (8.66). Notice also that the curly bracket $\{\mathbf{k} \to \mathbf{k}', \mathbf{n} \to \mathbf{n}', \lambda \to \lambda'\}^*$ in Eq. (8.68) has combined with the first one to give a norm squared, and the factor 1/4 is due to Eq. (8.70).

From Eq. (8.80), one finds C_ℓ by averaging over the $2\ell+1$ possible values of m according to Eq. (8.61). Thereby, the sum over m cancels with $\delta_{m,\pm 1}$ in Eq. (8.77) and one obtains the final expression

$$C_\ell = \frac{2}{\pi} \ell(\ell+1) \int_0^\infty \mathrm{d}k\, k^2 |\Delta_\ell(k)|^2,$$
$$\Delta_\ell(k) = -\frac{1}{\sqrt{2}} \left(\hat{\vartheta}_+ + i\hat{\vartheta}_\times\right) \frac{j_\ell[k(\eta_R - \eta_E)]}{k(\eta_R - \eta_E)} + \int_R^E \mathrm{d}\eta \frac{1}{\sqrt{2}} \left(\hat{\sigma}'_+ + i\hat{\sigma}'_\times\right) \frac{j_\ell[k(\eta_R - \eta)]}{k(\eta_R - \eta)}, \tag{8.81}$$

where $\hat{\vartheta}_\lambda = \hat{\vartheta}_\lambda(\eta_E, k)$, $\hat{\sigma}'_\lambda = \hat{\sigma}_\lambda(\eta, k)$, and the prime denotes a derivative with respect to η.

We give an alternative form for $\Delta_\ell(k)$ which might be useful in some cases. It is obtained by partial integration and using

$$(\ell-1)\frac{j_{\ell-1}}{x} - (\ell+2)\frac{j_{\ell+1}}{x} = j'_{\ell+1}(x) + j'_{\ell-1}(x). \tag{8.82}$$

It reads

$$\Delta_\ell(k) = -\frac{1}{\sqrt{2}} \Big((\hat{\vartheta}_+ + \hat{\sigma}_+) + i(\hat{\vartheta}_\times + \hat{\sigma}_\times) \Big) \frac{j_\ell[k(\eta_R - \eta_E)]}{k(\eta_R - \eta_E)}$$
$$+ \frac{k}{2\ell+1} \int_R^E \mathrm{d}\eta \frac{1}{\sqrt{2}} \left(\hat{\sigma}_+ + i\hat{\sigma}_\times\right) \left\{ j'_{\ell+1}[k(\eta_R - \eta)] + j'_{\ell-1}[k(\eta_R - \eta)] \right\}. \tag{8.83}$$

Here, the first term involves the vorticity $\hat{\vartheta}_\lambda + \hat{\sigma}_\lambda$, and the amplitude $\hat{\sigma}_\lambda$ appears without derivative.

Chapter 9

CMB anisotropies from vector perturbations in the bulk (article)

This chapter consists of the article 'CMB anisotropies from vector perturbations in the bulk', see Ref. [132].

It is also available under http://lanl.arXiv.org/abs/hep-th/0307100.

In this article, the internal coordinates on the brane are denoted by y^μ instead of σ^μ, in order to avoid confusion with the gauge invariant vector perturbation on the brane, σ_i.

CMB anisotropies from vector perturbations in the bulk

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The vector perturbations induced on the brane by gravitational waves propagating in the bulk are studied in a cosmological framework. Cosmic expansion arises from the brane motion in a non compact Z_2-symmetric five-dimensional anti-de Sitter space-time. By solving the vector perturbation equations in the bulk, for generic initial conditions, we find that they give rise to growing modes on the brane in the Friedmann-Lemaître era. Among these modes, we exhibit a class of normalizable perturbations, which are exponentially growing with respect to conformal time on the brane. The presence of these modes is strongly constrained by the current observations of the cosmic microwave background (CMB). We estimate the anisotropies they induce in the CMB, and derive quantitative constraints on the allowed amplitude of their primordial spectrum.

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9.1 Introduction

The idea that our universe may have more than three spatial dimensions has been originally introduced by Nordström [119], Kaluza [87] and Klein [94]. The fact that superstring theory, the most promising candidate for a theory of quantum gravity, is consistent only in ten space-time dimensions (11 dimensions for M theory) has led to a revival of these ideas [124, 125, 77]. It has also been found that string theories naturally predict lower dimensional "branes" to which fermions and gauge particles are confined, while gravitons (and the dilaton) propagate in the bulk [8, 126, 105]. Such brane worlds have been studied in a phenomenological way already before the discovery that they are actually realized in string theory [3, 136].

Recently it has been emphasized that relatively large extra-dimensions (with typical length $L \simeq \mu$m) can solve the hierarchy problem: the effective four-dimensional and the fundamental D-dimensional Newton constant are related by $G_4 \propto G_D/L^n$. Thus G_4 can become very small if the fundamental Planck mass is of the order of the electroweak scale. Here n denotes the number of extra-dimensions [12, 13, 9, 128]. It has also been shown that extra-dimensions may even be infinite if the geometry contains a so-called "warp factor" [127].

The size of the extra-dimensions is constrained by the requirement of recovering usual four-dimensional Einstein gravity on the brane, at least on scales tested by experiments [102, 155, 7]. Models with either a small Planck mass in the bulk [12, 13, 9], or with non compact warped extra-dimensions [128, 127], have been shown to lead to an acceptable cosmological phenomenology on the brane [19, 44, 43, 142, 60, 107, 135], with or without Z_2 symmetry in the bulk [40, 17, 41]. Explicit cosmological scenarii leading to a nearly Friedmann-Lemaître universe at late times can be realized on a 3–brane at rest in a dynamical bulk [18, 159] or, alternatively, on a brane moving in an anti-de Sitter bulk [98, 81]. It has been shown that both approaches are actually equivalent [116].

One can also describe brane worlds as topological defects in the bulk [23, 71, 11, 118, 66]. This is equivalent to the geometrical approach in the gravity sector [133], while it admits an explicit mechanism to confine matter and gauge fields on the brane [133, 130, 15, 56, 49, 55, 48, 50, 120, 67, 4, 117]. Depending on the underlying theory, the stability studies of these defects have shown that dynamical instabilities may appear on the brane when there are more than one non compact extra-dimensions [65, 88, 123], whereas this is not the case for a five-dimensional bulk [68], provided that a fine-tuning between the model parameters is fixed [21].

The next step is now to derive observational consequences of brane world cosmological models, *e.g.* the anisotropies of the cosmic microwave background (CMB). To that end, a lot of work has recently been invested to derive gauge invariant perturbation theory in brane worlds with one co-dimension [138, 100, 156, 32, 33]. Again, the perturbation equations can be derived when the brane is at rest [129], or when it is moving in a perturbed anti-de Sitter space-time [116, 114, 113, 115, 47, 46]. Whatever the approach chosen, the perturbation equations are quite cumbersome and it is difficult to extract interesting physical consequences analytically. Also the numerical treatment is much harder than in usual four-dimensional perturbation theory, since it involves partial differential equations.

Nevertheless, it is useful to derive some simple physical consequences of perturbation theory for brane worlds before performing intensive numerical studies. This has been done for tensor perturbations on the brane in a very phenomenological way in Ref. [101] or on a more fundamental level in Ref. [62]. Tensor modes in the bulk which induce scalar perturbations on the brane have been studied in Ref. [54] and it was found that they lead to important constraints for brane worlds.

In this article we consider a brane world in a five-dimensional bulk where cosmology is induced by the motion of a 3-brane in AdS_5 . The bulk perturbation equations are considered without bulk sources and describe gravity waves in the bulk. The present work concentrates on the part of these gravity waves which results in vector perturbations on the brane.

For the sake of clarity, we first recall how cosmology on the brane can be obtained via the junction conditions, particularly emphasizing how Z_2 symmetry is implemented [19, 44, 43, 142, 60, 107, 135]. After rederiving the bulk perturbation equations for the vector components in terms of gauge invariant variables [116, 114, 113, 115, 47], we analytically find the most general solutions for arbitrary initial conditions. The time evolution of the induced vector perturbations on the brane is then derived using the perturbed junction conditions. The main result of the paper is that vector perturbations in the bulk generically give rise to vector perturbations on the brane which grow either as a power law or even exponentially with respect to conformal time. This behavior essentially differs from the usual decay of vector modes in standard four-dimensional cosmology, and may lead to observable effects of extra-dimensions in the CMB.

The outline of the paper is the following: in the next section, the cosmological brane world model obtained by the moving brane in an anti-de Sitter bulk is briefly recalled. In Sec. 9.3 we set up the vector perturbation equations and solve them in the bulk. In Sec. 9.4 the induced perturbations on the brane are derived and compared to those in four-dimensional cosmology, while Sec. 9.5 deals with the consequences of these new results on CMB anisotropies. The resulting new constraints for viable brane worlds are discussed in the conclusion.

9.2 Background

As mentioned in the introduction, our universe is considered to be a 3-brane embedded in five-dimensional anti-de Sitter space-time

$$\mathrm{d}s_5^2 = G_{AB}\mathrm{d}x^A\mathrm{d}x^B = \frac{r^2}{L^2}\left(-\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) + \frac{L^2}{r^2}\mathrm{d}r^2. \tag{9.1}$$

The capital Latin indices A, B run from 0 to 4 and the flat spatial indexes i, j from 1 to 3. Anti-de Sitter space-time is a solution of Einstein's equations with a negative cosmological constant Λ

$$\mathcal{G}_{AB} + \Lambda G_{AB} = 0, \tag{9.2}$$

provided that the curvature radius L satisfies

$$L^2 = -\frac{6}{\Lambda}. \tag{9.3}$$

Another coordinate system for anti-de Sitter space can be defined by the coordinates transformation $r^2/L^2 = \mathrm{e}^{-2\varrho/L}$. Then the metric takes the form

$$\mathrm{d}s_5^2 = G_{AB}\mathrm{d}x^A\mathrm{d}x^B = \mathrm{e}^{-2\varrho/L}\left(-\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) + \mathrm{d}\varrho^2, \tag{9.4}$$

which is often used in brane world models.

9.2.1 Embedding and motion of the brane

The position of the brane in the AdS_5 bulk is given by

$$x^M = X^M(y^\mu), \tag{9.5}$$

where X^M are embedding functions depending on the internal brane coordinates y^μ ($\mu = 0, \cdots, 3$). Using the reparametrization invariance on the brane, we choose $x^i = X^i = y^i$. The other embedding functions are written

$$X^0 = t_{\rm b}(\tau), \quad X^4 = r_{\rm b}(\tau), \tag{9.6}$$

where $\tau \equiv y^0$ denotes cosmic time on the brane. Since we want to describe a homogeneous and isotropic brane, X^0 as well as X^4 are required to be independent of the spatial coordinates y^i. The four tangent vectors to the brane are given by

$$e^M_\mu \partial_M = \frac{\partial X^M}{\partial y^\mu}\partial_M, \tag{9.7}$$

and the unit space-like normal 1-form n_M is defined (up to a sign) by the orthogonality and normalization conditions

$$n_M e^M_\mu = 0, \quad G^{AB} n_A n_B = 1. \tag{9.8}$$

Adopting the sign convention that n points in the direction in which the brane is moving (growing $r_{\rm b}$ for an expanding universe), one finds using

$$e^0_\tau = \dot{t_{\rm b}}, \quad e^4_\tau = \dot{r_{\rm b}}, \quad e^i_j = \delta^i_j, \tag{9.9}$$

the components of the normal

$$n_0 = -\dot{r_{\rm b}}, \quad n_4 = \dot{t_{\rm b}}, \quad n_i = 0. \tag{9.10}$$

The other components are vanishing, and the dot denotes differentiation with respect to the brane time τ.

This embedding ensures that the induced metric on the brane describes a spatially flat homogeneous and isotropic universe,

$$ds^2_{\rm b} = g_{\mu\nu} dy^\mu dy^\nu = -d\tau^2 + a^2(\tau)\delta_{ij} dy^i dy^j, \tag{9.11}$$

where $a(\tau)$ is the usual scale factor, and $g_{\mu\nu}$ is the pull-back of the bulk metric onto the brane

$$g_{\mu\nu} = G_{AB} e^A_\mu e^B_\nu, \tag{9.12}$$

(see *e.g.* [38, 131]). The first fundamental form q_{AB} is now defined by

$$q^{AB} = g^{\mu\nu} e^A_\mu e^B_\nu, \tag{9.13}$$

i.e. the push-forward of the inverse of the induced metric tensor [39, 38]. One can also define an orthogonal projector onto the brane which can be expressed in terms of the normal 1-form

$$\perp_{AB} = n_A n_B = G_{AB} - q_{AB}, \tag{9.14}$$

in the case of only one codimension.

Upon inserting the equations (9.1), (9.10) and (9.13) into the above equation, one finds a parametric form for the brane trajectory [98, 81, 115, 47]

$$\begin{aligned} r_{\rm b}(\tau) &= a(\tau)L, \\ \dot{t_{\rm b}}(\tau) &= \frac{1}{a}\sqrt{1+L^2H^2}, \end{aligned} \tag{9.15}$$

where $H = \dot{a}/a$ denotes the Hubble parameter on the brane. Alternatively, this result can be obtained by comparing expression (9.12) with the Friedmann metric (9.11).

Therefore, the unperturbed motion induces a cosmological expansion on the 3-brane if $r_{\rm b}$ is growing with $t_{\rm b}$.

9.2.2 Extrinsic curvature and unperturbed junction conditions

The cosmological evolution on the brane is found by the Lanczos–Sen–Darmois–Israel junction conditions[1]. They relate the jump of the extrinsic curvature across the brane to its surface energy-momentum content [99, 141, 45, 82]. The extrinsic curvature tensor projected on the brane can be expressed in terms of the tangent and normal vectors as

$$k_{\mu\nu} = -e^A_\mu e^B_\nu \nabla_A n_B = -\frac{1}{2} e^A_\mu e^B_\nu \mathcal{L}_n G_{AB}. \tag{9.16}$$

Here ∇ denotes the covariant derivative with respect to the bulk metric, and $\mathcal{L}_n$ is the five-dimensional Lie-derivative in the direction of the unit normal on the brane. With the sign choice in Eq. (9.16), the junction conditions read [111]

$$k^>_{\mu\nu} - k^<_{\mu\nu} = \kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3} S\, g_{\mu\nu} \right) \equiv \kappa_5^2 \widehat{S}_{\mu\nu}, \tag{9.17}$$

where $S_{\mu\nu}$ is the energy momentum tensor on the brane with trace S, and

$$\kappa_5^2 \equiv 6\pi^2 G_5 = \frac{1}{M_5^3}, \tag{9.18}$$

where M_5 and G_5 are the five-dimensional (fundamental) Planck mass and Newton constant, respectively. The superscripts "$>$" and "$<$" stand for the bulk sides with $r > r_{\rm b}$ and $r < r_{\rm b}$. As already noticed, the brane normal vector n^M points into the direction of increasing r [see Eq. (9.10)]. Eq. (9.17) is usually referred to as second junction condition. The first junction condition simply states that the first fundamental form (9.13) is continuous across the brane.

In general, there is a force acting on the brane which is due to its curvature in the higher dimensional geometry. It is given by the contraction of the brane energy momentum tensor with the average of the extrinsic curvature on both sides of the brane [17]

$$S^{\mu\nu}\left(k^>_{\mu\nu} + k^<_{\mu\nu}\right) = 2f. \tag{9.19}$$

[1]In the following, they will be simply referred to as "junction conditions".

This force f, normal to the brane, is exerted by the asymmetry of the bulk with respect to the brane [38, 17]. In this paper, we consider only the case in which the bulk is Z_2-symmetric across the brane, hence $f = 0$. In this case the motion of the brane is caused by the stress energy tensor of the brane itself which is exactly the cosmological situation we have in mind.

From Eqs. (9.10), (9.11), (9.15) and (9.16), noting that the extrinsic curvature can be expressed purely in terms of the internal brane coordinates [47, 115], one has

$$k_{\mu\nu} = -\frac{1}{2}\left[G_{AB}\left(e^A_\mu \partial_\nu n^B + e^A_\nu \partial_\mu n^B\right) + e^A_\mu e^B_\nu n^C G_{AB,C}\right] . \tag{9.20}$$

A short computation shows that the non vanishing components of the extrinsic curvature tensor are

$$\begin{aligned} k_{\tau\tau} &= \frac{1 + L^2H^2 + \ell^2\dot{H}}{L\sqrt{1+L^2H^2}}, \\ k_{ij} &= -\frac{a^2}{L}\sqrt{1+L^2H^2}\delta_{ij}. \end{aligned} \tag{9.21}$$

It is clear, that the extrinsic curvature evaluated at some brane position $r_{\rm b}$ does not jump if the presence of the brane does not modify anti-de Sitter space. Like in the Randall–Sundrum (RS) model [127], in order to accommodate cosmology, the bulk space-time structure is modified by gluing the mirror symmetric of anti-de Sitter space on one side of the brane onto the other [116]. There are two possibilities: one can keep the "$r > r_{\rm b}$" side and replace the "$r < r_{\rm b}$" side to get

$$k^>_{\mu\nu} = k_{\mu\nu}, \quad k^<_{\mu\nu} = -k_{\mu\nu}, \tag{9.22}$$

where $k_{\mu\nu}$ is given by Eq. (9.21). Conversely, keeping the $r < r_{\rm b}$ side leads to

$$k^>_{\mu\nu} = -k_{\mu\nu}, \quad k^<_{\mu\nu} = k_{\mu\nu}. \tag{9.23}$$

Note that both cases verify the force equation (9.19). From the time and space components of the junction conditions (9.17) one obtains, respectively

$$\pm\frac{1 + L^2H^2 + \ell^2\dot{H}}{L\sqrt{1+L^2H^2}} = \frac{1}{2}\kappa_5^2\left(P+\rho\right) - \frac{1}{6}\kappa_5^2\left(\rho+\rho_T\right), \tag{9.24}$$

$$\pm\frac{\sqrt{1+L^2H^2}}{L} = -\frac{1}{6}\kappa_5^2\left(\rho+\rho_T\right). \tag{9.25}$$

Here the brane stress tensor is assumed to be that of a cosmological fluid plus a pure tension ρ_T, *i.e.*

$$S_{\mu\nu} = \left(P+\rho\right)u_\mu u_\nu + Pg_{\mu\nu} - \rho_T g_{\mu\nu}, \tag{9.26}$$

ρ and P being the usual energy density and pressure on the brane, and u^μ the comoving four-velocity. The "$\pm$" signs in Eqs. (9.24) and (9.25) are obtained by keeping, respectively, the $r > r_{\rm b}$, or $r < r_{\rm b}$, side of the bulk. In order to allow for a positive total brane energy density, $\rho + \rho_T$, we have to keep the $r < r_\mathrm{b}$ side and glue it symmetrically on the $r > r_\mathrm{b}$ one[2]. In the trivial static ($H = 0$) case this construction reproduces the Randall–Sundrum II [127] solution with warp factor $\exp(-|\varrho|/L)$, for $-\infty < \varrho < \infty$ if we choose $r_\mathrm{b} = L =$ constant. In our coordinates, we just have $0 < r \leq r_\mathrm{b}$ on either side of the brane, and the bulk is now described by two copies of the "bulk behind the brane". Even if r only takes values inside a finite interval, and even though the volume of the extra dimension,

$$V = 2\int_0^{r_\mathrm{b}} \sqrt{|G|}\,\mathrm{d}r = \frac{r_\mathrm{b}}{2}\left(\frac{r_\mathrm{b}}{L}\right)^3, \tag{9.27}$$

is finite, the bulk is semi-compact and its spectrum of perturbation modes has no gap (like in the RS model).

From Eqs. (9.24) and (9.25), one can check that energy conservation on the brane is verified

$$\dot{\rho} + 3H\left(P + \rho\right) = 0. \tag{9.28}$$

Solving Eq. (9.25) for the Hubble parameter yields

$$H^2 = \frac{\kappa_5^4 \rho_T}{18}\rho\left(1 + \frac{\rho}{2\rho_T}\right) + \frac{\kappa_5^4}{36}\rho_T^2 - \frac{1}{L^2}. \tag{9.29}$$

At "low energies", $|\rho/\rho_T| \ll 1$, the usual Friedmann equation is recovered provided the fine-tuning condition

$$\frac{\kappa_5^4}{36}\rho_T^2 = \frac{1}{L^2} \tag{9.30}$$

is satisfied. The four-dimensional Newton constant is then given by

$$\kappa_4^2 \equiv 8\pi G_4 = \frac{\kappa_5^4 \rho_T}{6}. \tag{9.31}$$

Thus a positive tension is required to get a positive effective four-dimensional Newton constant. Note also that low energy means $\tau^2 \sim H^{-2} \gg L^2$. In the Friedmann-Lemaître era, the solution of Eq. (9.29) reads

$$\begin{aligned} H &\simeq H_0\left(\frac{a}{a_0}\right)^{-3(1+w)/2}, \\ \dot{H} &\simeq -\frac{3}{2}(1+w)H_0^2\left(\frac{a}{a_0}\right)^{-3(1+w)} = -\frac{3}{2}(1+w)H^2, \end{aligned} \tag{9.32}$$

for a cosmological equation of state $P = w\rho$ with constant w. The parameters H_0 and a_0 refer, respectively, to the Hubble parameter and the scale factor today. For the matter era we have $w = 0$, and during the radiation era $w = 1/3$.

[2] Note that we obtain the same result as in Ref. [47]: a positive brane tension for an expanding universe is obtained by keeping the anti-de Sitter side which is "behind the expanding brane with respect to its motion".

9.3 Gauge invariant perturbation equations in the bulk

A general perturbation in the bulk can be decomposed into "3-scalar", "3-vector" and "3-tensor" parts which are irreducible components under the group of isometries (of the unperturbed space time) $SO(3) \times E_3$, the group of three dimensional rotations and translations. In this paper we restrict ourselves to 3-vector perturbations[3] and consider an "empty bulk", *i.e.* the case where there are no sources in the bulk except a negative cosmological constant. With respect to the bulk, and its four spatial dimensions, only bulk gravity waves are therefore considered since they are the only modes present when the energy momentum tensor is not perturbed. It is well known (see *e.g.* Ref. [129]) that gravity waves in $4+1$ dimensions have five degrees of freedom which can be decomposed with respect to their spin in $3+1$ dimensions into a spin 2 field, the ordinary graviton, a spin 1 field, often called the gravi-photon and into a spin 0 field, the gravi-scalar. In this work we study the evolution of the gravi-photon in the background described in the previous section.

After setting up our notations, we find the gauge invariant vector perturbation variables in the bulk and write down the perturbed Einstein equations. We derive analytic solutions for all vector modes in the bulk.

9.3.1 Bulk perturbation variables

Considering only vector perturbations in the bulk, the five dimensional perturbed metric can be parameterized as

$$\begin{aligned} \mathrm{d}\tilde{s}_5^2 = &-\frac{r^2}{L^2}\mathrm{d}t^2 + \frac{r^2}{L^2}\left(\delta_{ij} + \nabla_i E_j + \nabla_j E_i\right)\mathrm{d}x^i\mathrm{d}x^j \\ &+ \frac{L^2}{r^2}\mathrm{d}r^2 + 2B_i\frac{r^2}{L^2}\mathrm{d}t\mathrm{d}x^i + 2C_i\mathrm{d}x^i\mathrm{d}r, \end{aligned} \tag{9.33}$$

where ∇_i denotes the connection in the three dimensional subspace of constant t and constant r. Assuming this space to be flat one has $\nabla_i = \partial_i$. The quantities E^i, B^i, and C^i are divergenceless vectors *i.e.* $\partial_i E^i = \partial_i B^i = \partial_i C^i = 0$.

As long as we want to solve for the vector perturbations in the bulk only, the presence of the brane is not yet relevant. Later it will appear as a boundary condition for the bulk perturbations via the junction conditions as will be discussed in paragraph 9.4.3.

Under a linearized vector type coordinate transformation in the bulk, $x^M \to x^M + \varepsilon^M$, with $\varepsilon_M = (0, \varepsilon_i, 0)$, the perturbation variables defined above transform

[3] The prefix "3-" will be dropped in what follows, and the term "vector" will be always applied here for spin 1 with respect to the surfaces of constant t and r.

as

$$
\begin{aligned}
E_i &\to E_i + \frac{L^2}{r^2}\varepsilon_i, \\
B_i &\to B_i + \frac{L^2}{r^2}\partial_t\varepsilon_i, \\
C_i &\to C_i + \partial_r\varepsilon_i - \frac{2}{r}\varepsilon_i.
\end{aligned}
\tag{9.34}
$$

As expected for three divergenceless vector variables and one divergenceless vector type gauge transformation, there remain four degrees of freedom which are described by the two gauge invariant vectors

$$\Sigma_i = B_i - \partial_t E_i, \tag{9.35}$$

$$\Xi_i = C_i - \frac{r^2}{L^2}\partial_r E_i. \tag{9.36}$$

Note that in the gauge $E_i = 0$ these gauge-invariant variables simply become B_i and C_i respectively.

9.3.2 Bulk perturbation equations and solutions

A somewhat cumbersome derivation of the Einstein tensor from the metric (9.33) to first order in the perturbations leads to the following vector perturbation equations,

$$\partial_t\Sigma - \frac{L}{r}\partial_r\left(\frac{r^3}{L^3}\Xi\right) = 0, \tag{9.37}$$

$$\frac{r^4}{L^2}\partial_r^2\Sigma + 5\frac{r^3}{L^2}\partial_r\Sigma - L^2\partial_t^2\Sigma + L^2\Delta\Sigma = 0, \tag{9.38}$$

$$\frac{r^4}{L^2}\partial_r^2\left(\frac{r^3}{L^3}\Xi\right) - \frac{r^3}{L^2}\partial_r\left(\frac{r^3}{L^3}\Xi\right) - L^2\partial_t^2\left(\frac{r^3}{L^3}\Xi\right) + L^2\Delta\left(\frac{r^3}{L^3}\Xi\right) = 0, \tag{9.39}$$

where Δ denotes the spatial Laplacian, *i.e.*

$$\Delta = \delta^{ij}\partial_i\partial_j, \tag{9.40}$$

and the spatial index on Σ and Ξ has been omitted. One can check that these equations are consistent, *e.g.* with the master function approach of Ref. [113].

A complete set of solutions for these equations can easily be found by Fourier transforming with respect to x^i, and making the separation ansatz:

$$
\begin{aligned}
\Sigma(t, r, \mathbf{k}) &= \Sigma_{\mathrm{T}}(t, \mathbf{k})\Sigma_{\mathrm{R}}(r, \mathbf{k}), & (9.41)\\
\Xi(t, r, \mathbf{k}) &= \Xi_{\mathrm{T}}(t, \mathbf{k})\Xi_{\mathrm{R}}(r, \mathbf{k}). & (9.42)
\end{aligned}
$$

The most general solution is then a linear combination of such elementary modes.

Eq. (9.38) splits into two ordinary differential equations for Σ_{T} and Σ_{R},

$$r^4 \frac{\partial_r^2 \Sigma_{\mathrm{R}}}{\Sigma_{\mathrm{R}}} + 5r^3 \frac{\partial_r \Sigma_{\mathrm{R}}}{\Sigma_{\mathrm{R}}} = \pm L^4 \Omega^2, \tag{9.43}$$

$$\frac{\partial_t^2 \Sigma_{\mathrm{T}}}{\Sigma_{\mathrm{T}}} + k^2 = \pm \Omega^2, \tag{9.44}$$

where k is the spatial wave number, and $\pm\Omega^2$ the separation constant having the dimension of an inverse length squared. The frequency Ω represents the rate of change of Σ_{R} at $r \sim L$, while the rate of change of Σ_{T} is $\sqrt{|\Omega^2 \mp k^2|}$. From the four-dimensional point of view, $-\Omega^2$ can also be interpreted as the mass m^2 of the mode so that $\pm\Omega^2 = -m^2$. The signs in Eqs. (9.43) and (9.44) come from the choice $\Omega^2 \geq 0$. Eq. (9.43) is a Bessel differential equation of order two for the "$-$" sign and a modified Bessel equation of order two for the "$+$" sign [1], while Eq. (9.44) exhibits oscillatory or exponential behavior in bulk time. From Eq. (9.39), similar equations are derived for $\Xi_{\mathrm{T}}(t, \mathbf{k})$ and $\Xi_{\mathrm{R}}(r, \mathbf{k})$. This time, the radial function is given by Bessel functions of order one. The constraint equation (9.37) ensures that the separation constant $\pm\Omega^2$ is the same for both vectors and it also determines their relative amplitude. The general solution of Eqs. (9.37) to (9.39) is a superposition of modes $\Omega, \mathbf{k}$ which are given by

$$\Sigma \propto \begin{cases} \dfrac{L^2}{r^2} \mathrm{K}_2\!\left(\dfrac{L^2\Omega}{r}\right) \mathrm{e}^{\pm t\sqrt{\Omega^2 - k^2}} \\ \dfrac{L^2}{r^2} \mathrm{I}_2\!\left(\dfrac{L^2\Omega}{r}\right) \mathrm{e}^{\pm t\sqrt{\Omega^2 - k^2}} \\ \dfrac{L^2}{r^2} \mathrm{J}_2\!\left(\dfrac{L^2\Omega}{r}\right) \mathrm{e}^{\pm it\sqrt{\Omega^2 + k^2}} \\ \dfrac{L^2}{r^2} \mathrm{Y}_2\!\left(\dfrac{L^2\Omega}{r}\right) \mathrm{e}^{\pm it\sqrt{\Omega^2 + k^2}}, \end{cases} \tag{9.45}$$

$$\Xi \propto \begin{cases} \pm\sqrt{1 - k^2/\Omega^2}\,\dfrac{L^2}{r^2} \mathrm{K}_1\!\left(\dfrac{L^2\Omega}{r}\right) \mathrm{e}^{\pm t\sqrt{\Omega^2 - k^2}} \\ \pm\sqrt{1 - k^2/\Omega^2}\,\dfrac{L^2}{r^2} \mathrm{I}_1\!\left(\dfrac{L^2\Omega}{r}\right) \mathrm{e}^{\pm t\sqrt{\Omega^2 - k^2}} \\ \pm\sqrt{1 + k^2/\Omega^2}\,\dfrac{L^2}{r^2} \mathrm{J}_1\!\left(\dfrac{L^2\Omega}{r}\right) i\mathrm{e}^{\pm it\sqrt{\Omega^2 + k^2}} \\ \pm\sqrt{1 + k^2/\Omega^2}\,\dfrac{L^2}{r^2} \mathrm{Y}_1\!\left(\dfrac{L^2\Omega}{r}\right) i\mathrm{e}^{\pm it\sqrt{\Omega^2 + k^2}}. \end{cases} \tag{9.46}$$

Here K_p and I_p are the modified Bessel functions of order p while J_p and Y_p are the ordinary ones. The $\pm$ signs in Eqs. (9.45) and (9.46) correspond to the two linearly independent solutions of Eq. (9.44), whereas the sign of the separation constant determines the kind of Bessel functions: $+\Omega^2$ (or $m^2 < 0$) for the modified Bessel function K and I; $-\Omega^2$ (or $m^2 > 0$) for the ordinary Bessel functions J and Y. In general, each of these modes[4] can be multiplied by a proportionality coefficient which depends on the wave vector $\mathbf{k}$ and Ω. Eq. (9.37) ensures that this coefficient is the same for Σ and Ξ. Furthermore, notice that if $\Omega^2 > k^2$ the K– and I–modes can have an exponentially growing behavior, whereas for $\Omega^2 < k^2$ one sets $\sqrt{\Omega^2 - k^2} = i\sqrt{|\Omega^2 - k^2|}$ such that the modes become oscillatory. The J– and Y–modes are always oscillating.

For a given perturbation mode to be physically acceptable one has to require that, at some initial time $t_{\rm in}$, the perturbations are small for all values $0 < r \leq r_{\rm b}(t_{\rm in})$, compared to the background. To check that, we use the limiting forms of the Bessel functions [1]. For large arguments, the ordinary Bessel functions behave as

$$\begin{aligned}
\mathrm{J}_p(x) &\underset{x\to\infty}{\sim} \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{2}p - \frac{\pi}{4}\right),\\
\mathrm{Y}_p(x) &\underset{x\to\infty}{\sim} \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{2}p - \frac{\pi}{4}\right),
\end{aligned} \tag{9.47}$$

while the modified Bessel functions grow or decrease exponentially

$$\begin{aligned}
\mathrm{K}_p(x) &\underset{x\to\infty}{\sim} \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x},\\
\mathrm{I}_p(x) &\underset{x\to\infty}{\sim} \frac{1}{\sqrt{2\pi x}}\mathrm{e}^{x}.
\end{aligned} \tag{9.48}$$

Therefore, in Eqs. (9.45) and (9.46) all modes, except for the K–mode, diverge as $r \to 0$. Hence the only regular modes are

$$\Sigma = A(\mathbf{k},\Omega)\frac{L^2}{r^2}\mathrm{K}_2\left(\frac{L^2}{r}\Omega\right)\mathrm{e}^{\pm t\sqrt{\Omega^2 - k^2}}, \tag{9.49}$$

$$\Xi = \pm A(\mathbf{k},\Omega)\sqrt{1 - \frac{k^2}{\Omega^2}}\,\frac{L^2}{r^2}\mathrm{K}_1\left(\frac{L^2}{r}\Omega\right)\mathrm{e}^{\pm t\sqrt{\Omega^2 - k^2}}, \tag{9.50}$$

where the amplitude $A(\mathbf{k},\Omega)$ is determined by the initial conditions and carries an implicit spatial index. For small wave numbers $k^2 < \Omega^2$ the growing solution rapidly dominates, whereas for large wave numbers $k^2 > \Omega^2$ both solutions are comparable and oscillating in time. It is easy to see that the K–mode is also normalizable in the sense that

$$\begin{aligned}
\int_0^{r_{\rm b}} \sqrt{|G|}|\Sigma|^2\mathrm{d}r &\propto \int_0^{r_{\rm b}} \frac{1}{r}\left[\mathrm{K}_2\left(\frac{L^2}{r}\Omega\right)\right]^2 \mathrm{d}r < \infty,\\
\int_0^{r_{\rm b}} \sqrt{|G|}|\Xi|^2\mathrm{d}r &\propto \int_0^{r_{\rm b}} \frac{1}{r}\left[\mathrm{K}_1\left(\frac{L^2}{r}\Omega\right)\right]^2 \mathrm{d}r < \infty.
\end{aligned} \tag{9.51}$$

Note that also the J–modes and Y–modes are normalizable. One might view this integrability condition as a requirement to insure finiteness of the energy of

[4] In the following, they will be labeled by the kind of Bessel function they involve, *e.g.* "K–mode", "I–mode" *etc.*

these modes. This suggests that the J– and Y–modes could also be excited by some physical process. Indeed, from Eq. (9.4), their divergence for $r \to 0$ can be recast in terms of the ϱ coordinate, with $\varrho \to \infty$. Expressed in terms of ϱ, the integrability condition (9.51) ensures that the J– and Y–modes are well defined in the Dirac sense, and thus that a superposition of them may represent physical perturbations[5] [151].

Let us briefly discuss also the zero-mode $\Omega = 0$. The solution of Eqs. (9.37), (9.38) and (9.39) are then

$$\Sigma = A\mathrm{e}^{\pm ikt}, \tag{9.52}$$

$$\Xi = \pm ikL\mathrm{e}^{\pm ikt}\left[\frac{A}{2}\frac{L}{r} + B\left(\frac{L}{r}\right)^3\right]. \tag{9.53}$$

These solutions (which can also be obtained from Eqs. (9.45) and (9.46) in the limit $\Omega \to 0$) diverge for $r \to 0$ but the A–mode is normalizable in the sense that the integrals defined in Eq. (9.51) converge.

Clearly the most intriguing solutions are the K–modes, especially for values of the separation constant verifying $\Omega^2 > k^2$. Then, if present, these fluctuations soon dominate the others in the bulk. The fact that $m^2 = -\Omega^2 < 0$ in this case implies that the K–modes are tachyonic modes, and it is thus not surprising that they may generate instabilities.

Before we go on, let us just note, that all these solutions are also valid solutions of the bulk vector perturbation equations in the RS model. In their original work [127], Randall and Sundrum have obtained very similar equations (we used somewhat different variables). However, they considered only the solutions with $m^2 = +\Omega^2 > 0$ and therefore did not find the growing K–modes. As we shall see in the next section, this choice is justified when one considers boundary conditions which do not allow for any anisotropic stresses on the brane. This is indeed well motivated as far as cosmology is not concerned. A more detailed discussion of the relevance of these modes for the RS model is given in appendix 9.7.1.

In our cosmological framework however, if there is no physical argument which forbids these modes, they have to be taken seriously since they represent solutions of the perturbations equations which are small at very early times and grow exponentially with respect to bulk time. Note that this instability is linked to the particular bulk structure considered here where the brane lies at one boundary of the space-time. In the full AdS_5, $0 < r < \infty$, the K–modes are clearly not normalizable since $\mathrm{K}_p(L^2\Omega/r)$ diverges for $r \to \infty$.

At last, one may hope that the K–modes are never generated. However, during any bulk inflationary phase which leads to the production of $4+1$ dimensional gravity waves, as we shall see now, if the anisotropic stresses on the brane do not vanish identically the K–modes are perfectly admissible solutions. For a given inflationary model, it should be also possible to calculate the spectrum of fluctuations, $|A(\mathbf{k},\Omega)|^2$.

[5] This is of course not the case for the I–modes.

At this stage, the perturbation modes have been only derived in the bulk. In the next section, we shall determine the induced perturbations on the brane using the perturbed junction conditions.

9.4 The induced perturbations on the brane

9.4.1 Brane perturbation variables

Since we are interested in vector perturbations on the brane induced by those in the bulk, we parameterize the perturbed induced metric as

$$\begin{aligned} \mathrm{d}\tilde{s}_{\mathrm{b}}^2 &= \tilde{g}_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu \\ &= -\mathrm{d}\tau^2 + 2ab_i \mathrm{d}\tau \mathrm{d}y^i \\ &+ a^2 \left(\delta_{ij} + \nabla_i e_j + \nabla_j e_i\right) \mathrm{d}y^i \mathrm{d}y^j, \end{aligned} \tag{9.54}$$

where e^i and b^i are divergence free vectors. The junction conditions which relate the bulk perturbation variables to the perturbations of the brane can be written in terms of gauge invariant variables. Under an infinitesimal transformation $y^\mu \to y^\mu + \xi^\mu$, where $\xi_\mu = (0, a^2\xi_i)$, we have

$$\begin{aligned} e_i &\to e_i + \xi_i, \\ b_i &\to b_i + a\dot{\xi}_i. \end{aligned} \tag{9.55}$$

Here the dot is the derivative with respect to the brane time τ and ξ_i is a divergence free vector field. Hence the gauge invariant vector perturbation is [52]

$$\sigma_i = b_i - a\dot{e}_i. \tag{9.56}$$

This variable fully describes the vector metric perturbations on the brane.

The brane energy momentum tensor $S_{\mu\nu}$ given in Eq. (9.26) has also to be perturbed. As we shall see, the junction conditions (together with Z_2 symmetry) do in general require a perturbed energy momentum tensor on the brane. Since we only consider vector perturbations $\delta\rho = \delta P = 0$. However, the perturbed four-velocity of the perfect fluid does contain a vector part $\tilde{u}^\mu = u^\mu + \delta u^\mu$, with

$$\delta u^\mu = \begin{pmatrix} 0 \\ \dfrac{v^i}{a} \end{pmatrix}, \tag{9.57}$$

and where v^i is divergenceless. Under $y^\mu \to y^\mu + \xi^\mu$,

$$v_i \to v_i - a\dot{\xi}_i, \tag{9.58}$$

where $v_i \equiv \delta_{ij} v^i$. A gauge invariant perturbed velocity can therefore be defined as

$$\vartheta_i = v_i + a\dot{e}_i. \tag{9.59}$$

In addition, the anisotropic stresses contain a vector component denoted π_i. Since the corresponding background quantity vanishes, this variable is gauge invariant according to the Stewart–Walker lemma [146].

In summary, there are three gauge invariant brane perturbation variables. We shall use the combinations

$$\begin{aligned} \sigma_i &= b_i - a\dot{e}_i, \\ \vartheta_i &= v_i + a\dot{e}_i, \\ \pi_i&. \end{aligned} \tag{9.60}$$

To apply the junction conditions we need to determine the perturbations of the reduced energy momentum tensor defined in Eq. (9.17). In terms of our gauge invariant quantities they read

$$\delta\widehat{S}_{\tau\tau} = 0, \tag{9.61}$$

$$\delta\widehat{S}_{\tau i} = -a\left(P + \frac{2}{3}\rho - \frac{1}{3}\rho_T\right)\sigma_i - a\left(P+\rho\right)\vartheta_i, \tag{9.62}$$

$$\delta\widehat{S}_{ij} = a^2 P\left(\partial_i\pi_j + \partial_j\pi_i\right). \tag{9.63}$$

9.4.2 Perturbed induced metric and extrinsic curvature

We now express the perturbed induced metric, and the perturbed extrinsic curvature in terms of the bulk perturbation variables [47]. In principle there are two contributions to the brane perturbations: perturbations of the bulk geometry as well as perturbations of the brane position. A bulk perturbed quantity has then to be evaluated at the perturbed brane position [see Eq. (9.6)]. Using reparametrization invariance on the brane [47], the perturbed embedding can be described in terms of a single variable Υ,

$$\tilde{X}^M = X^M + \Upsilon n^M, \tag{9.64}$$

where all quantities are functions of the brane coordinates y^μ. Since Υ is a scalar perturbation it does not play a role in our treatment, and we can consider only the perturbations δG_{AB} due to the perturbed bulk geometry evaluated at the unperturbed brane position. The induced metric perturbation is then given by

$$\delta g_{\mu\nu} = \tilde{g}_{\mu\nu} - g_{\mu\nu} = e^A_\mu e^B_\nu \delta G_{AB}. \tag{9.65}$$

From Eqs. (9.33), (9.35), (9.36) and (9.65) one finds in the gauge $E_i = 0$

$$\begin{aligned} \delta g_{\tau\tau} &= 0, \\ \delta g_{\tau i} &= a\sqrt{1+L^2H^2}\Sigma_i + aLH\Xi_i, \\ \delta g_{ij} &= \delta G_{ij} = 0. \end{aligned} \tag{9.66}$$

The time component vanishes as it is a pure scalar, and the purely spatial components can be set to zero without loss of generality by gauge fixing ($E_i = e_i = 0$).

In the same way, perturbing Eq. (9.20), and making use of Eqs. (9.7), (9.8) in order to derive the perturbed normal vector, leads to (again we use the gauge $E_i = 0$)

$$\delta k_{\tau\tau} = 0, \tag{9.67}$$

$$\begin{aligned}\delta k_{\tau i} &= \frac{1}{2}\partial_t \Xi_i - \frac{1}{2}a^2\partial_r\Sigma_i - aH\sqrt{1+L^2H^2}\Xi_i \\ &\quad - \frac{a}{L}\left(1+L^2H^2\right)\Sigma_i,\end{aligned} \tag{9.68}$$

$$\begin{aligned}\delta k_{ij} &= \frac{1}{2}aLH\left(\partial_i\Sigma_j + \partial_j\Sigma_i\right) \\ &\quad + \frac{1}{2}a\sqrt{1+L^2H^2}\left(\partial_i\Xi_j + \partial_j\Xi_i\right),\end{aligned} \tag{9.69}$$

where $\delta k_{\mu\nu} = \tilde{k}_{\mu\nu} - k_{\mu\nu}$, and all bulk quantities have to be evaluated at the brane position. In the derivation we have also used that on the brane $\partial_\mu = e^A_\mu \partial_A$.

9.4.3 Perturbed junction conditions and solutions

The first junction condition requires the first fundamental form q_{AB} to be continuous across the brane. Therefore, the components of the induced metric (9.54) are given by the explicit expressions (9.66). This leads to the following relations

$$\begin{aligned} e_i &= E_i, \\ b_i &= \sqrt{1+L^2H^2}B_i + LHC_i, \end{aligned} \tag{9.70}$$

where the bulk quantities have to be evaluated at the brane position $(t_{\rm b}, r_{\rm b})$. For $\sigma_i = b_i - a\dot{e}_i$ we use

$$\begin{aligned} a\dot{e}_i &= a\left(\dot{t_{\rm b}}\partial_t E_i + \dot{r_{\rm b}}\partial_r E_i\right) \\ &= \sqrt{1+L^2H^2}\partial_t E_i + a^2 LH\partial_r E_i. \end{aligned} \tag{9.71}$$

Together with Eqs. (9.35) and (9.36) this gives

$$\sigma_i = \sqrt{1+L^2H^2}\Sigma_i + LH\Xi_i. \tag{9.72}$$

The equations corresponding to the second junction condition are obtained by perturbing Eq. (9.17) (using Z_2 symmetry, $k^{>}_{\mu\nu} = -k^{<}_{\mu\nu} = -k_{\mu\nu}$) and inserting the expressions (9.68), (9.69) for the perturbed extrinsic curvature tensor, with Eqs. (9.62), (9.63) for the perturbed energy-momentum tensor on the brane. After some algebra one obtains for the $0i$ and the ij components, respectively

$$\frac{2L\dot{H}}{\sqrt{1+L^2H^2}}a\left(\sigma_i + \vartheta_i\right) = a^2\partial_r\Sigma_i - \partial_t\Xi_i, \tag{9.73}$$

$$\kappa_5^2 aP\pi_i = -LH\Sigma_i - \sqrt{1+L^2H^2}\Xi_i, \tag{9.74}$$

where we have used the unperturbed junction conditions, Eqs. (9.24) and (9.25), and the fact that on the brane $\partial_\mu = e^A_\mu \partial_A$.

In the RS model one has $H = \dot{H} = 0$ and the requirement that the anisotropic stresses vanish identically. We show in appendix 9.7.1 that the well-known results of Refs. [127] and [151] are recovered in this limit.

Hence, if by some mechanism, like *e.g.* bulk inflation, gravity waves are produced in the bulk, their vector parts Σ_i and Ξ_i will induce vector metric perturbations σ_i on the brane according to Eq. (9.72). The vorticity $\sigma_i + \vartheta_i$ and anisotropic stresses π_i on the brane define boundary conditions for the bulk variables according to Eqs. (9.73) and (9.74). In general, the time evolution of π_i may be given by some additional matter equation, like *e.g.* the Boltzmann equation or some dissipation equation which usually depends also on the metric perturbations. It is interesting to note that for generic initial conditions in the bulk, the amplitude of the K–mode does not vanish, which means that the anisotropic stresses on the brane may grow exponentially. At late time (for $LH \ll 1$) Eq. (9.74) reduces to $\kappa_5^2 a P \pi_i = -\Xi_i$. A generally covariant equation of motion for π_i must be compatible with this behavior since it is a simple consequence of the 5-dimensional Einstein equations for a certain choice of initial conditions.

In the following we do not want to specify a particular mechanism which generates Σ_i and Ξ_i, and just assume they have been produced with some spectrum given by $A(\mathbf{k}, \Omega)$. In usual 4-dimensional cosmology it is well-known that vector perturbations decay. Therefore, in ordinary 4-dimensional inflationary scenarios they are not considered. Only if they are continuously re-generated like, *e.g.* in models with topological defects (see *e.g.* Ref. [51]), vector modes affect CMB anisotropies. Here the situation is different since the modes considered are either exponentially growing or oscillating with respect to bulk time. Therefore, we expect the behavior of vector perturbations to be very different from the usual 4-dimensional results even in the absence of K–modes.

In the following, we assume that the boundary and initial conditions are such the $\pi_i(t_{\rm in}) \not\equiv 0$. They therefore allow for K–mode contributions. Clearly, if this happens it leads to exponential growth of σ_i and π_i. However, before concluding about the viability of these modes, one has to check if they have observable consequences on the brane. Indeed, anisotropic stresses are often very small (*e.g.* of second order only) and one may therefore hope that the initial amplitudes of the K–modes are also very small and do not lead to destructives effects, at least on time scales equal to the age of the universe. By estimating the induced CMB anisotropies, we show in the next section that this is not the case.

9.5 CMB anisotropies

To calculate the CMB anisotropies from the vector perturbations induced by bulk gravity waves, the relevant quantities are σ and $\vartheta + \sigma$ given in terms of the bulk variables by Eqs. (9.72) and (9.73). Inserting the solutions (9.49) and (9.50) for the K–mode into (9.72) and (9.73) yields

$$\begin{aligned}\sigma(t_{\rm b},\mathbf{k}) = A(\mathbf{k},\Omega)\Bigg[&\sqrt{1+L^2H^2}\,\frac{1}{a^2}\mathrm{K}_2\!\left(\frac{L\Omega}{a}\right)\\ &\pm LH\sqrt{1-\frac{k^2}{\Omega^2}}\,\frac{1}{a^2}\mathrm{K}_1\!\left(\frac{L\Omega}{a}\right)\Bigg]\mathrm{e}^{\pm t_{\rm b}\sqrt{\Omega^2-k^2}},\end{aligned} \tag{9.75}$$

and

$$(\sigma+\vartheta)\,(t_{\rm b},\mathbf{k}) = A(\mathbf{k},\Omega)\frac{k^2}{\Omega^2}\frac{\sqrt{1+L^2H^2}}{2L^2\dot{H}}\frac{L\Omega}{a}\frac{1}{a^2}\mathrm{K}_1\!\left(\frac{L\Omega}{a}\right)\mathrm{e}^{\pm t_{\rm b}\sqrt{\Omega^2-k^2}}, \tag{9.76}$$

where again we have omitted the spatial index i on σ, ϑ and A. Similar equations can be obtained for the J– and Y–modes by replacing, in Eqs. (9.75) and (9.76), the modified Bessel functions by the ordinary ones, plus the transformations: $-k^2 \to k^2$ and $\pm \to \pm i$.

These equations are still written in bulk time $t_{\rm b}$ which is related to the conformal time η on the brane by

$$\mathrm{d}t_{\rm b} = \sqrt{1+L^2H^2}\,\mathrm{d}\eta. \tag{9.77}$$

Therefore, at sufficiently late time $L^2H^2 \ll 1$ such that $\mathrm{d}t_{\rm b} \simeq \mathrm{d}\eta$. Note that L is the size of the extra-dimension which must be smaller than micrometers while H^{-1} is the Hubble scale which is larger than 10^5 light years at times later than recombination which are of interest for CMB anisotropies.

As a result, the growing or oscillating behavior in bulk time carries over to conformal time. Moreover there are additional time dependent terms in Eqs. (9.75) and (9.76) with respect to Eqs. (9.49) and (9.50) due to the motion of the brane. As can be seen from Eqs. (9.75) and (9.76), the modes evolve quite differently for different values of their physical bulk wave number Ω/a. In the limit $\Omega/a \ll 1/L$ and for $\Omega^2 > k^2$, the growing K–modes behave like

$$\begin{aligned}\sigma &\sim \frac{2A}{(\Omega L)^2}\,\mathrm{e}^{\eta\sqrt{\Omega^2-k^2}},\\ \sigma+\vartheta &\sim \frac{A}{(\Omega L)^2}\frac{k^2}{2a^2\dot{H}}\,\mathrm{e}^{\eta\sqrt{\Omega^2-k^2}},\end{aligned} \tag{9.78}$$

where use has been made of $L^2H^2 \ll 1$, and of the limiting forms of Bessel function for small arguments [1]

$$\mathrm{K}_p(x) \underset{x\to 0}{\sim} \frac{1}{2}\Gamma(p)\left(\frac{2}{x}\right)^p. \tag{9.79}$$

In the same way, from Eq. (9.48), the K–modes verifying $\Omega/a \gg 1/L$ reduce to

$$\begin{aligned}\sigma &\sim \frac{A}{(\Omega L)^2}\mathrm{e}^{\eta\sqrt{\Omega^2-k^2}}\sqrt{\frac{\pi}{2}}\left(\frac{\Omega L}{a}\right)^{3/2}\mathrm{e}^{-\Omega L/a},\\ \sigma+\vartheta &\sim \frac{A}{(\Omega L)^2}\frac{k^2}{2a^2\dot{H}}\,\mathrm{e}^{\eta\sqrt{\Omega^2-k^2}}\sqrt{\frac{\pi}{2}}\left(\frac{\Omega L}{a}\right)^{1/2}\mathrm{e}^{-\Omega L/a}.\end{aligned} \tag{9.80}$$

They are exponentially damped compared to the former [see Eq. (9.78)]. As a result, the main contribution of the K–mode vector perturbations comes from the modes with a physical wave number Ω/a smaller than the energy scale $1/L$ associated with the extra-dimension. As the universe expands, a mode with fixed value Ω remains relatively small as long as the exponents in Eq. (9.80) satisfy

$$\frac{\Omega}{a}L - \eta\sqrt{\Omega^2 - k^2} \simeq \frac{1}{a}\left(\Omega L - \tau\sqrt{\Omega^2 - k^2}\right) > 0. \tag{9.81}$$

When this inequality is violated, for $k \ll \Omega$ this is soon after $\tau \sim L$, the mode starts growing exponentially. The time $\tau \sim L$ also corresponds to the initial time at which the evolution of the universe starts to become Friedmannian.

In the same way, one can derive the behavior of the J– and Y–modes on the brane for physical bulk wave numbers greater or smaller than the size of the extra-dimension. This time, the exponentially growing terms are replaced by oscillatory ones, and the ordinary Bessel functions are approximated by (see Eq. (9.47) and Ref. [1])

$$\begin{aligned} \mathrm{J}_{p}(x) &\underset{x\to 0}{\sim} \frac{1}{\Gamma(p+1)}\left(\frac{x}{2}\right)^{p}, \\ \mathrm{Y}_{p}(x) &\underset{x\to 0}{\sim} -\frac{1}{\pi}\Gamma(p)\left(\frac{2}{x}\right)^{p}. \end{aligned} \tag{9.82}$$

From Eqs. (9.47) and (9.82), the equivalents of Eqs. (9.75) and (9.76) for J– and Y–modes can be shown to oscillate always. From Eq. (9.32), their amplitude is found to decay like $a^{-3/2}$ in the short wavelength limit $\Omega/a \gg 1/L$. In the long wavelength limit $\Omega/a \ll 1/L$, the amplitude of the Y–mode stays constant whereas the J–mode decreases as a^{-4}.

The vorticity is also found to oscillate in conformal time. This time, the amplitude of the long wavelength Y–modes always grows as a^{3w+1} while for the J–modes it behaves like a^{3w-1}. Finally, in the short wavelength limit, both Y and J vorticity modes grow like $a^{3w+1/2}$.

Whatever the kind of physical vector perturbation modes excited in the bulk, we have shown that there always exist bulk wave numbers Ω that give rise to growing vector perturbations on the brane. Although the J– and Y–modes generate vorticity growing like a power law of the scale factor, they can be, in a first approximation, neglected compared to the K–modes which grows like an exponential of the conformal time. We therefore now concentrate on the K–modes and derive constraints on their initial amplitude $A(\mathbf{k},\Omega)$ by estimating the CMB anisotropies they induce.

In order to determine the temperature fluctuations in the CMB due to vector perturbations on the brane, we have to calculate how a photon emitted on the last scattering surface travels through the perturbed geometry (9.54). A receiver today (with conformal time η_R) therefore measures different microwave background temperatures $T(\eta_R, n^i)$ for incident photons coming from different directions n^i. With the conventions and formulae in the paragraphs 8.3.3 and 8.3.4 the vector-type temperature fluctuations read

$$\begin{aligned}\frac{\delta T_{\mathrm{R}}(n^i)}{\bar{T}_{\mathrm{R}}} &= n^i\,(\sigma_i + \vartheta_i)|_{\mathrm{E}}^{\mathrm{R}} + \int_{\mathrm{E}}^{\mathrm{R}} \frac{\partial \sigma_i}{\partial x^j} n^i n^j \mathrm{d}\lambda, \\ &= -n^i \vartheta_i(\eta_{\mathrm{E}}) + \int_{\mathrm{E}}^{\mathrm{R}} \sigma_i' n^i \mathrm{d}\eta,\end{aligned} \tag{9.83}$$

where λ denotes the affine parameter along the photon trajectory and the prime is a derivative with respect to conformal time η. The "R" and "E" index refer to the time of photon reception (today) and emission at recombination. For the second equality we have used

$$\frac{\mathrm{d}\sigma_i}{\mathrm{d}\lambda} = \sigma_i' - n^j \frac{\partial \sigma_i}{\partial x^j}, \tag{9.84}$$

where $-n^i$ is the direction of the photon momentum. We have also neglected the contribution from the upper boundary, "R", in the first term since it simply gives rise to a dipole term. The first term in Eq. (9.83) is a Doppler shift, and the second is known as integrated Sachs-Wolfe effect. To determine the angular CMB power spectrum C_ℓ, we apply the total angular momentum formalism developed by Hu and White [80]. According to this, a vector perturbation $\mathbf{v}$ is decomposed as

$$\mathbf{v} = \mathbf{e}^+ v^+ + \mathbf{e}^\times v^\times, \tag{9.85}$$

where $\mathbf{e}^{+,\times}$ are defined so that ($\mathbf{e}^+$, $\mathbf{e}^\times$, and $\hat{\mathbf{k}} = \mathbf{k}/k$) form a righthanded orthonormal system. Using this decomposition for ϑ_i and σ_i, one obtains the angular CMB perturbation spectrum C_ℓ via

$$C_\ell = \frac{2}{\pi}\ell(\ell+1)\int_0^\infty k^2 \langle|\Delta_\ell(k)|^2\rangle \mathrm{d}k, \tag{9.86}$$

where

$$\begin{aligned}\Delta_\ell(k) &= -\vartheta^+(\eta_{\mathrm{E}}, k)\frac{j_\ell(k\eta_0 - k\eta_{\mathrm{E}})}{k\eta_0 - k\eta_{\mathrm{E}}} \\ &\quad + \int_{\eta_{\mathrm{E}}}^{\eta_0} \sigma^{+\prime}(\eta, k)\frac{j_\ell(k\eta_0 - k\eta)}{k\eta_0 - k\eta}\mathrm{d}\eta.\end{aligned} \tag{9.87}$$

In Eq. (9.87) we have assumed that the process which generates the fluctuations has no preferred handedness so that $\langle|\sigma^+|^2\rangle = \langle|\sigma^\times|^2\rangle$ as well as $\langle|\vartheta^+|^2\rangle = \langle|\vartheta^\times|^2\rangle$. Omitting the $+$ and $\times$ superscripts, we can take into account the other mode simply by a factor 2. Furthermore, we have redefined $\eta_0 \equiv \eta_R$.

As shown in the previous section, the main contribution of the K–modes comes from those having long wavelengths $a/\Omega \gg L$, and $k < \Omega$. In the following, only these modes will be considered. Since they are growing exponentially in η, the integrated Sachs-Wolfe contribution will dominate and we concentrate on it in what follows. A more rigorous justification is given in appendix 9.7.2. Inserting the limiting form (9.78) for σ in Eq. (9.87) gives

$$\begin{aligned} \Delta_\ell(\bar{k}) \simeq 2A_{_0}\Omega^n\bar{k}^n \mathrm{e}^{\varpi_0\sqrt{1-\bar{k}^2}}\sqrt{\frac{1}{\bar{k}^2}-1} \\ \times \int_0^{x_{\mathrm{E}}} \frac{j_\ell(x)}{x}\mathrm{e}^{-x\sqrt{1/\bar{k}^2-1}}\mathrm{d}x, \end{aligned} \tag{9.88}$$

where a simple power law ansatz has been chosen for the primordial amplitude

$$\sqrt{\langle |A(\mathbf{k},\Omega)|^2\rangle} = A_{_0}(\Omega)\Omega^2 L^2 k^n. \tag{9.89}$$

A dimensionless wave number $\bar{k}$, and conformal time ϖ, have also been introduced as

$$\bar{k} = \frac{k}{\Omega}, \quad \varpi = \eta\Omega, \tag{9.90}$$

in order to measure their physical counterparts in units of the bulk wavelengths. The condition $k < \Omega$ now becomes $\bar{k} < 1$. The integration over η in the integrated Sachs-Wolfe term is transformed into an integration over the dimensionless variable x defined by

$$x = k\left(\eta_{_0} - \eta\right) = \bar{k}\left(\varpi_{_0} - \varpi\right), \tag{9.91}$$

the subscript "0" refers to the present time. Note that $x_{_\mathrm{E}} = k\left(\eta_{_0} - \eta_{_\mathrm{E}}\right) \simeq k\eta_{_0}$.

By observing the CMB, one may naturally expect that the perturbations with physical wavelength greater than the horizon size today have almost no effect. In terms of our parameters, this means that the main contribution in the C_ℓ comes from the modes verifying $\Omega/a_{_0} > H_{_0}$, hence $\varpi_{_0} > 1$.

In appendix 9.7.2, we derive a crude approximation for the angular power spectrum induced by the exponentially growing K–modes, in a range a little more constrained than the one previously motivated, namely

$$\ell_{\max} H_{_0} < \frac{\Omega}{a_{_0}} < \frac{L^{-1}}{1+z_{_\mathrm{E}}}, \tag{9.92}$$

where $z_{_\mathrm{E}}$ is the redshift at photon emission which is taken to coincide with recombination, $z_{_\mathrm{E}} \simeq 10^3$. In order to simplify the calculation, we do not want the transition between the damped K–modes ($\Omega/a > L^{-1}$) and the exponentially growing ones ($\Omega/a < L^{-1}$) to occur between the last scattering surface and today. This requirement leads to the upper limit of Eq. (9.92). Moreover, in order to derive the C_ℓ, we perform an expansion with respect to a parameter $\ell/\varpi_{_0}$ assumed small, and $\ell_{\max}$ refers to the multipole at which this approximation breaks down. The lower limit in Eq. (9.92) comes from this approximation. Using the values $L \simeq 10^{-3}\,\mathrm{mm}$, $H_{_0}^{-1} \simeq 10^{29}\,\mathrm{mm}$, $\ell_{\max} \simeq 10^3$, and $z_{_\mathrm{E}} \simeq 10^3$, one finds

$$10^{-26}\,\mathrm{mm}^{-1} < \frac{\Omega}{a_{_0}} < 1\,\mathrm{mm}^{-1}. \tag{9.93}$$

The corresponding allowed range for the parameter ϖ_{0} becomes [see Eq. (9.116)]

$$10^3 < \varpi_{0} < 10^{29}. \tag{9.94}$$

Clearly the detailed peak structure on the CMB anisotropy spectrum would have been different if we had taken into account the oscillatory parts ($k > \Omega$) of the K–modes, as well as the Y– and J–modes, but here we are only interested in estimating an order of magnitude bound. As detailed in appendix 9.7.2, for a scale invariant initial spectrum, *i.e.* $n = -3/2$, we obtain

$$\frac{\ell(\ell+1)}{2\pi} C_\ell \gtrsim (A_{0} e^{\varpi_0})^2 \frac{e^{-\ell}}{\ell^{7/2}} \left(\frac{\ell}{\varpi_{0}}\right)^{\ell-1}. \tag{9.95}$$

From current observations of the CMB anisotropies, the left hand side of this expression is about 10^{-10}, and for $\ell \simeq 10$, one gets

$$A_{0}(\Omega) \lesssim \frac{e^{-\left[\varpi_0 - 5\ln(\varpi_0)\right]}}{10^5}. \tag{9.96}$$

From Eq. (9.116) and (9.94), one finds that the primordial amplitude of these modes must satisfy

$$A_{0}(\Omega) < e^{-10^3}, \text{ for } \Omega/a_{0} \simeq 10^{-26}\text{mm}^{-1} \tag{9.97}$$

and, more dramatically,

$$A_{0}(\Omega) < e^{-10^{29}}, \text{ for } \Omega/a_{0} \simeq 1\text{mm}^{-1} \tag{9.98}$$

for the short wavelength modes. As expected, the perturbations with wavelength closer to the horizon today (smaller values of Ω) are less constrained than smaller wavelengths [see Eq. (9.97)]. Moreover, one may expect that the bound (9.98) is no longer valid for $\Omega/a_{0} > L^{-1}/(1 + z_{\mathrm{E}})$ since the modes in Eq. (9.80) start to contribute. However, the present results concern more than 20 orders of magnitude for the physical bulk wave numbers Ω/a_{0}, and show that the exhibited modes are actually very dangerous for the brane world model we are interested in.

It seems that the only way to avoid these constraints is to find a physical mechanism forbidding any excitation of these modes.

9.6 Conclusion

In this paper we have shown that vector perturbations in the bulk generically lead to growing vector perturbations on the brane in the Friedmann-Lemaître era. This behavior radically differs from the usual one in four-dimensional cosmology, where vector modes decay like a^{-2} whatever the initial conditions.

Among the growing modes, we have identified so called K–modes which are perfectly normalizable and lead to exponentially growing vector perturbations on the brane with respect to conformal time. By means of a rough estimate of the CMB anisotropies induced by these perturbations, we have shown that they are severely incompatible with a homogeneous and isotropic universe; they light up a fire in the microwave sky, unless their primordial amplitude is extremely small.

No particular mechanism for the generation of these modes has been specified. However, one expects that bulk inflation leads to gravitational waves in the bulk which do generically contain them. Even if they are not generated directly, they should be induced in the bulk by second order effects. Usually, these effects are too small to have any physical consequences, but here they would largely suffice due to the exponential growth of the K–modes [see Eqs. (9.97) and (9.98)]. This second order induction seems very difficult to prevent in the models discussed here.

It is interesting to note that this result is also linked to the presence of a non compact extra-dimension which allows a continuum of bulk frequencies Ω. A closer examination of Eq. (9.44) shows that the mode $\Omega = 0$, admits only J– and Y–mode behaviors. In a compact space, provided the first quantized value of Ω is sufficiently large, one could expect the exponentially growing K–modes to be never excited by low energy physical processes. Another more speculative way to dispose of them could be to consider their causal structure: as we have noticed before, the modes with separation constant $+\Omega^2$ are tachyons of mass $-\Omega^2$ from the four-dimensional point of view. From the five-dimensional point of view, these are not "propagating modes", but "brane-modes" which decay into the fifth dimension with penetration depth $d = L^2\Omega$.

In a more basic theory, which goes beyond our classical relativistic approach, these modes may thus not be allowed at all.

Finally, we want to retain that even if the K–modes can be eliminated in some way, the growing behavior of the Y– and J–modes remains. Although their power law growth is not as critical as the exponential growth of the K–modes, they should have significant effects on the CMB anisotropies. Indeed, they lead to amplified oscillating vector perturbations which are entirely absent in four-dimensional cosmology.

We therefore conclude that, if no physical mechanism forbids the generation of the discussed vector modes with time dependence $\propto \exp(\eta\sqrt{\Omega^2 - k^2})$, anti-de Sitter infinitly thin brane worlds, with non compact extra-dimension, cannot reasonably lead to a homogeneous and isotropic expanding universe.

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9.7 Appendix

9.7.1 Comparison with the Randall-Sundrum model

As already mentioned in the text, if the brane is at rest ($H = 0$) at $r_{\rm b} = L$, our model reduces to the RS II model. One may ask therefore, quite naturally, why has our dangerous K–mode never been discussed in the context of RS II? In this appendix we address this question.

First of all, the bulk solutions Σ and Ξ for vector perturbations of AdS$_5$ with a brane, remain valid. The solutions with $m^2 = -\Omega^2 < 0$ have, however not been discussed in the literature so far. Also, when constructing the Green's function [63, 127, 151], these solutions have not been considered. As we shall see now, for most problems that is most probably very reasonable.

In the RS II model one considers perturbations which do not require anisotropic stresses on the brane, $\pi_i = 0$. Eq. (9.74) then reduces to

$$\Xi(r = L, t, \Omega, k) = 0 \ , \tag{9.99}$$

such that Ξ has to vanish on the brane. We insert this into a general solution of the form

$$\begin{aligned}
\Xi(r,t,\Omega,k) &= \pm i\sqrt{1+k^2/\Omega^2}\,\frac{L^2}{r^2}\mathrm{e}^{\pm it\sqrt{\Omega^2+k^2}} \\
&\quad\times \left[A\mathrm{J}_1\!\left(\frac{L^2\Omega}{r}\right) + B\mathrm{Y}_1\!\left(\frac{L^2\Omega}{r}\right)\right], \qquad (9.100)\\
&\quad \text{for}\quad m^2 = \Omega^2 > 0,\\
\Xi(r,t,\Omega,k) &= \pm \sqrt{1-k^2/\Omega^2}\,\frac{L^2}{r^2}\mathrm{e}^{\pm t\sqrt{\Omega^2-k^2}} \\
&\quad\times \left[C\mathrm{K}_1\!\left(\frac{L^2\Omega}{r}\right) + D\mathrm{I}_1\!\left(\frac{L^2\Omega}{r}\right)\right], \qquad (9.101)\\
&\quad \text{for}\quad m^2 = -\Omega^2 < 0\ .
\end{aligned}$$

The boundary condition (9.99) then implies

$$B = -A\frac{\mathrm{J}_1(L\Omega)}{\mathrm{Y}_1(L\Omega)}, \qquad \text{for}\quad m^2 = \Omega^2 > 0, \tag{9.102}$$

$$D = -C\frac{\mathrm{K}_1(L\Omega)}{\mathrm{I}_1(L\Omega)}, \qquad \text{for}\quad m^2 = -\Omega^2 < 0. \tag{9.103}$$

Eq. (9.102) is exactly the relation which has also been found in Ref. [127], while Eq. (9.103) is new. However, if the solution is not allowed to grow exponentially when approaching the Cauchy horizon $r \to 0$, one has to require $D = 0$, which implies $C = 0$ since K_1 has no zeros. With this physically sensible condition (see Ref. [151]), we can discard these solutions. Nevertheless, in cases where the I–modes can be regularized, *e.g.* by compactification, presence of a second brane, the most general Green's function would include them. It is interesting to note that the calculation of the static potential of two masses M_1 and M_2 at distance x generated by the exchange of the zero-mode and the two continua of Kaluza–Klein modes with positive and with imaginary masses, simply leads to

$$\begin{aligned} V(x) &\sim G_4 \frac{M_1 M_2}{x}\left(1+\int_0^\infty mL^2 \mathrm{e}^{-mx}\mathrm{d}m - \int_0^\infty mL^2\mathrm{e}^{-imx}\mathrm{d}m\right) \\ &= G_4\frac{M_1M_2}{x}\left(1+\frac{2L^2}{x^2}\right) . \end{aligned} \tag{9.104}$$

The short distance modification hence deviates by a factor of 2 from the result of Ref. [127], if we include the tachyonic modes. One has to be aware of the fact that, like so often, the result is sensitive to the choice of the Green's functions.

Anyway, small initial perturbations of the RS solution which allow for small anisotropic stresses, so that the condition (9.103) does not need to be imposed, will in general contain a small K–mode which grows exponentially and renders the cosmological model unstable. It seems to us that this possibility has been overlooked in the literature so far.

We give a simple example which sketches the presence of this instability. We consider a $1+1$ dimensional Minkowski space-time, with orbifold-like spatial sections which can be identified with two copies of $y \geq 0$. The "brane" is represented by the point $y=0$ and the "bulk" by the two copies of $y>0$. For an initially small perturbation $f(y,t)$ in the bulk, which satisfies a hyperbolic wave equation, we want to analyze whether an instability can build up. We are looking for solutions of

$$\partial_t^2 f - \partial_y^2 f = 0, \tag{9.105}$$

with small initial data, say $f(t=0,y) \ll 1$ and $\partial_t f(t=0,y) \ll 1$ for all $y \geq 0$. By separation of variables one can find a complete set of solutions, $f = f_\pm(k)\exp[\pm ik(y\pm t)]$. For a sufficiently small value of $f_\pm(k)$ these solutions satisfy the initial conditions. These solutions oscillate in time; they have constant amplitude. However, there are other solutions, $f = g_\pm(k)\exp[\pm k(y\pm t)]$. Since the initial data has to be small, the solutions $\propto \exp(+ky)$ are not allowed. But the solutions $f = g_-\exp[-k(y-t)]$ have perfectly small initial data and they represent an exponential instability. If we fix the boundary conditions, setting $f(t,y=0)=0$, or $\partial_t f(t,y=0)=0$, this instability disappears, but if $f(t,y=0)$ is free, even a very small initial value $f(0,0) \ll 1$ can induce an exponential instability. Clearly, this leads also to an exponential growth of the boundary value $f(t,y=0)$.

If we give $f(0,y) = A\exp(-ky)$ and $\partial_t f(0,y) = kA\exp(-ky)$ as initial conditions, the function $f(t,y) = A\exp[k(t-y)]$ solves the equation and generates the growing exponential. If we would require, as an additional boundary condition that, *e.g.* the solutions at $y=0$ remain at least bounded, this mode would not be allowed and we would have to expand the initial data in terms of the oscillatory modes. However, it seems to us acausal to pose conditions of what is going to happen "on the brane" in the future. But mathematically, without any such "acausal" boundary conditions, the initial value problem is not well posed. This example is a simple analog of our instability. As long as anisotropic stresses vanish identically, only the J– and Y–modes are relevant. However, if the brane has arbitrarily small but non vanishing anisotropic stresses on which we do not want to impose any constraints for their future behavior, an exponential instability can build up. This is a rather unnatural behavior which may cast doubts on the RS realisation of brane worlds in the context of cosmological perturbation theory.

9.7.2 CMB angular power spectrum

We first present a crude and then a more sophisticated approximation for the C_ℓ power spectrum from the exponentially growing K–modes. As we shall see, at moderate values of $\ell \sim 10$–50, both lead to roughly the same bounds for the amplitudes which are also presented in the text.

Crude approximation

Here we start from Eq. (9.88). In the integral

$$\int_0^{x_{\rm E}} \frac{j_\ell(x)}{x} \mathrm{e}^{-x\sqrt{1/\bar{k}^2-1}} \mathrm{d}x, \tag{9.106}$$

we replace j_ℓ by its asymptotic expansion for small ℓ,

$$j_\ell(x) \simeq \left(\frac{x}{2}\right)^\ell \frac{\sqrt{\pi}}{2\Gamma(\ell+3/2)}. \tag{9.107}$$

This is a good approximation if either $x_{\rm E} \simeq k\eta_0 < \ell/2$ or $(\ell/2)(1/\bar{k}^2-1)^{1/2} > 1$. Since $k^2 < \Omega^2$, the first condition is always satisfied if the first of the two inequalities in (9.92) are fulfilled. The integral of x then gives

$$\langle|\Delta_\ell(\bar{k})|^2\rangle \simeq \frac{\pi A_0^2 \Omega^{2n} \bar{k}^{2n}}{2^{2\ell}\ell^3} \left(\frac{1}{\bar{k}^2} - 1\right)^{1-\ell} \mathrm{e}^{2\varpi_0\sqrt{1-\bar{k}^2}}. \tag{9.108}$$

Integrating over k, we must take into account that our approximation is only valid for $k < k_{\rm max} = (\Omega^2 - \eta_0^{-2})^{1/2}$. Since we integrate a positive quantity we certainly obtain a lower bound by integrating it only until $k_{\rm max}$. To simplify the integral we also make the transformation of variables $y = \sqrt{1-\bar{k}^2}$. With this and upon inserting our result (9.108) in Eq. (9.86), we obtain

$$\ell^2 C_\ell \gtrsim \frac{2\ell}{2^{2\ell}} A_0^2 \Omega^{2n+3} \int_{1/\varpi_0}^1 (1-y^2)^{n+\ell-1/2} y^{3-2\ell} \mathrm{e}^{2\varpi_0 y} \mathrm{d}y. \tag{9.109}$$

For $\ell \geq 2$, $y^{3-2\ell} \geq 1$ on the entire range of integration. Hence we have

$$\ell^2 C_\ell \gtrsim \frac{2\ell}{2^{2\ell}} A_0^2 \Omega^{2n+3} \int_0^1 (1-y^2)^{n+\ell-1/2} \mathrm{e}^{2\varpi_0 y} \mathrm{d}y. \tag{9.110}$$

This integral can be expressed in terms of modified Struve functions [1]. In the interesting range, $\varpi_0 \gg 1$, we have

$$\int_0^1 (1-y^2)^{n+\ell-1/2} \mathrm{e}^{2\varpi_0 y} \mathrm{d}y \simeq \frac{\Gamma(n+\ell+1/2)}{4\varpi_0^{n+\ell+1/2}} \mathrm{e}^{2\varpi_0}. \tag{9.111}$$

Inserting this result in Eq. (9.110) we finally obtain

$$\begin{aligned} \ell^2 C_\ell &\gtrsim \frac{\sqrt{2\pi}\sqrt{\ell}\,\mathrm{e}^{-\ell}}{2^{2\ell+1}} \left(\frac{\ell}{\varpi_0}\right)^{n+\ell+1/2} A_0^2 \Omega^{2n+3} \mathrm{e}^{2\varpi_0} \\ &\sim \frac{\sqrt{2\pi}\sqrt{\ell}\,\mathrm{e}^{-\ell}}{2^{2\ell+1}} \left(\frac{\ell}{\varpi_0}\right)^{\ell-1} A_0^2 \mathrm{e}^{2\varpi_0}, \end{aligned} \tag{9.112}$$

where we have used Stirling's formula for $\Gamma(\ell+n+1/2)$ and set $n=-3/2$ after the $\sim$ sign.

In the next paragraph we use a somewhat more sophisticated method which allows us to calculate also the Doppler contribution to the C_ℓ's. For the ISW effect this method gives

$$\ell^2 C_\ell \simeq \sqrt{\frac{2}{\pi}} \frac{\mathrm{e}^{-\ell}}{36\ell^{7/2}} \left(\frac{\ell}{\varpi_0}\right)^{\ell-1} A_0^2 \mathrm{e}^{2\varpi_0} \tag{9.113}$$

for $n=-3/2$. Until $\ell \sim 15$ the two approximations are in reasonable agreement and lead to the same prohibitive bounds for $A_0(\Omega)$. For $\ell > 15$, Eq. (9.113) becomes more stringent.

Sophisticated approximation

In Eq. (9.88) we have only considered the dominant contribution coming from the integrated Sachs-Wolfe effect. The general expression is obtained by inserting the solutions (9.78) for σ and ϑ in Eq. (9.87),

$$\begin{aligned} \Delta_\ell(\bar{k}) &= 2A_0\Omega^n \bar{k}^n \left(1-\frac{\bar{k}^2}{\bar{k}_\mathrm{E}^2}\right) \frac{j_\ell(\bar{k}\varpi_0 - \bar{k}\varpi_\mathrm{E})}{\bar{k}\varpi_0 - \bar{k}\varpi_\mathrm{E}} \mathrm{e}^{\varpi_\mathrm{E}\sqrt{1-\bar{k}^2}} \\ &\quad + 2A_0\Omega^n\bar{k}^n \mathrm{e}^{\varpi_0\sqrt{1-\bar{k}^2}} \sqrt{\frac{1}{\bar{k}^2}-1} \int_0^{x_\mathrm{E}} \frac{j_\ell(x)}{x} \mathrm{e}^{-x\sqrt{1/\bar{k}^2-1}} \mathrm{d}x. \end{aligned} \tag{9.114}$$

To derive the first term we have used Eq. (9.32) in the matter era. The parameter

$$\bar{k}_\mathrm{E}^2 = 6(1+z_\mathrm{E}) \left(\frac{H_0 a_0}{\Omega}\right)^2 \tag{9.115}$$

reflects the change in behavior of the modes, redshifted by z_E to the emission time, which are either outside or inside the horizon today. It is important to note that the parameter $H_0 a_0/\Omega$ completely determines the effect of the bulk vector perturbations on the CMB, together with the primordial amplitude A_0. Indeed, solving Eq. (9.29) in terms of conformal time, and using Eqs. (9.28) and (9.32), yields $\eta_0 \simeq 2/(a_0 H_0)$ in the Friedmann-Lemaître era. Thus

$$\varpi_0 \simeq 2\frac{\Omega/a_0}{H_0}, \quad \varpi_{\mathrm{E}} \simeq \frac{1}{1+z_{\mathrm{E}}}\frac{\Omega/a_0}{H_0}. \tag{9.116}$$

We now replace the spherical Bessel functions j_ℓ in the integrated Sachs-Wolfe term (ISW) using the relation [1]

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}\mathrm{J}_{\ell+1/2}(x). \tag{9.117}$$

In the ISW term the upper integration limit can be taken to be infinity as the contribution from x_E to infinity can be neglected provided $x\sqrt{1/\bar{k}^2-1} > 1$. This restriction is equivalent to $\bar{k}^2 < 1 - 1/\varpi_0$ which is verified for almost all values of $\bar{k}$ up to one, given that ϖ_0 varies in the assumed range (9.94). We remind that for the exponentially growing K–mode $k \leq \Omega$, and hence $0 \leq \bar{k} \leq 1$. This allows for the exact solution [70]

$$\int_0^\infty x^{-3/2}\mathrm{J}_{\ell+1/2}(x)\mathrm{e}^{-x\sqrt{1/\bar{k}^2-1}}\mathrm{d}x = \frac{\bar{k}^\ell}{2^{\ell+1/2}}\frac{\Gamma(\ell)}{\Gamma(\ell+3/2)}F\left(\frac{\ell}{2},\frac{\ell}{2}+1;\ell+\frac{3}{2};\bar{k}^2\right), \tag{9.118}$$

where F is the Gauss hypergeometric function. In regard to the subsequent integration over k we approximate F as follows. For small values of $\bar{k}$, F is nearly constant with value 1, at $\bar{k}=0$. As $\bar{k}\to 1$ the slope of F diverges and it cannot be Taylor expanded anymore. However, by means of the linear transformation formulas [1], F can be written as a combination of hypergeometric functions depending on $1-\bar{k}^2$

$$\begin{aligned} F\left(\frac{\ell}{2},\frac{\ell}{2}+1;\ell+\frac{3}{2};\bar{k}^2\right) &= \frac{\Gamma\left(\ell+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\ell}{2}+\frac{3}{2}\right)\Gamma\left(\frac{\ell}{2}+\frac{1}{2}\right)}F\left(\frac{\ell}{2},\frac{\ell}{2}+1;\frac{1}{2};1-\bar{k}^2\right) \\ &+ \sqrt{1-\bar{k}^2}\frac{\Gamma\left(\ell+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{\ell}{2}\right)\Gamma\left(\frac{\ell}{2}+1\right)}F\left(\frac{\ell}{2}+\frac{3}{2},\frac{\ell}{2}+\frac{1}{2};\frac{3}{2};1-\bar{k}^2\right). \end{aligned} \tag{9.119}$$

These in turn can be expanded around $1-\bar{k}^2=0$, and gives

$$F \underset{\bar{k}\to 1}{\sim} 2^{\ell+1/2}\left(1-\ell\sqrt{1-\bar{k}^2}\right). \tag{9.120}$$

These two approximations intersect at $\bar{k} = \sqrt{1-1/\ell^2}$. In this way, we can evaluate the mean value of F by integrating the two parts over the interval $[0,1]$. Thus, the hypergeometric function is replaced by

$$F\left(\frac{\ell}{2},\frac{\ell}{2}+1;\ell+\frac{3}{2};\bar{k}^2\right) \simeq \frac{2^{\ell+1/2}}{6\ell^2}. \tag{9.121}$$

Furthermore, the Gamma functions in (9.118) can be approximated using Stirling's formula [1]

$$\frac{\Gamma(\ell)}{\Gamma(\ell+3/2)} \simeq \frac{1}{\ell^{3/2}}. \tag{9.122}$$

Putting everything together and squaring Eq. (9.114) we obtain

$$\begin{aligned}
|\Delta_\ell(\bar{k})|^2 = 2\pi A_0^2 \bar{k}^{2n}\Omega^{2n} \Bigg\{ & \left(1-\frac{\bar{k}^2}{\bar{k}_{\rm E}^2}\right)^2 \frac{{\rm e}^{2\varpi_{\rm E}\sqrt{1-\bar{k}^2}}}{\bar{k}^3(\varpi_0-\varpi_{\rm E})^3}\left({\rm J}_{\ell+1/2}\left[\bar{k}(\varpi_0-\varpi_{\rm E})\right]\right)^2 \\
& + 2\left(1-\frac{\bar{k}^2}{\bar{k}_{\rm E}^2}\right)\frac{{\rm e}^{(\varpi_0+\varpi_{\rm E})\sqrt{1-\bar{k}^2}}}{\bar{k}^{3/2}(\varpi_0-\varpi_{\rm E})^{3/2}}{\rm J}_{\ell+1/2}\left[\bar{k}(\varpi_0-\varpi_{\rm E})\right]\frac{\bar{k}^{\ell-1}\sqrt{1-\bar{k}^2}}{6\ell^{7/2}} \\
& + {\rm e}^{2\varpi_0\sqrt{1-\bar{k}^2}}\frac{\bar{k}^{2(\ell-1)}(1-\bar{k}^2)}{36\ell^7}\Bigg\}.
\end{aligned} \tag{9.123}$$

The C_ℓ's are then found by integrating over all k-modes

$$\begin{aligned}
C_\ell &= \frac{2}{\pi}\ell(\ell+1)\Omega^3\int_0^1 \bar{k}^2|\Delta_\ell(\bar{k})|^2{\rm d}\bar{k} \\
&\equiv 4A_0^2\ell(\ell+1)\Omega^{2n+3}\left(C_\ell^{(1)}+C_\ell^{(2)}+C_\ell^{(3)}\right),
\end{aligned} \tag{9.124}$$

where the $C_\ell^{(i)}$ correspond to the three terms in Eq. (9.123). In the following we keep only the zero order terms in $\varpi_0/\varpi_{\rm E}$. From Eqs. (9.123), (9.124) one finds

$$C_\ell^{(1)} = \frac{1}{\varpi_0^3}\int_0^1 \bar{k}^{2n-1}\left(1-\frac{\bar{k}^2}{\bar{k}_{\rm E}^2}\right)^2 {\rm e}^{2\varpi_{\rm E}\sqrt{1-\bar{k}^2}}\left[{\rm J}_{\ell+1/2}\left(\bar{k}\varpi_0\right)\right]^2{\rm d}\bar{k}. \tag{9.125}$$

First, notice that if the argument is larger or smaller than the index, the Bessel functions are well approximated by their asymptotic expansions (9.47) and (9.82), respectively. Therefore, we split the $\bar{k}$-integral into two integrals over the intervals $[0,\bar{k}_\ell]$ and $[\bar{k}_\ell,1]$, in each of which the Bessel function is replaced by its limiting forms. The transition value $\bar{k}_\ell$ is given by $\bar{k}_\ell \simeq \ell/\varpi_0$. In the integral from $\bar{k}_\ell$ to 1, the $\sin^2(\bar{k}\varpi_0)$ is then replaced by its mean value 1/2 which is justified if the multiplying function varies much slower in $\bar{k}$ than the sine. To carry out the integration we make the substitution $y^2 = 1-\bar{k}^2$, and in order to simplify the notation we define the integral

$$\mathcal{I}(a,b,\nu) = \int_a^b y(1-y^2)^\nu {\rm e}^{2\varpi_{\rm E}y}{\rm d}y. \tag{9.126}$$

In this way we can write Eq. (9.125) in the form

$$\begin{aligned} C_\ell^{(1)} &= \frac{1}{\pi\varpi_0^4}\Big[\mathcal{I}(0,y_\ell,n-3/2) - \frac{2}{\bar{k}_{\rm E}^2}\mathcal{I}(0,y_\ell,n-1/2) + \frac{1}{\bar{k}_{\rm E}^4}\mathcal{I}(0,y_\ell,n+1/2)\Big] \\ &+ \frac{1}{\Gamma^2(\ell+3/2)}\left(\frac{\varpi_0}{2}\right)^{2\ell+1}\Big[\mathcal{I}(y_\ell,1,\ell+n-1/2) \\ &- \frac{2}{\bar{k}_{\rm E}^2}\mathcal{I}(y_\ell,1,\ell+n+1/2) + \frac{1}{\bar{k}_{\rm E}^4}\mathcal{I}(y_\ell,1,\ell+n+3/2)\Big]. \end{aligned} \tag{9.127}$$

Since $y_\ell = \sqrt{1-\bar{k}_\ell^2}$ is very close to one, and the integrand is continuous in the interval $[0,1]$, integrals of the form $\mathcal{I}(y_\ell,1,\nu)$ can be well approximated by the mean value

$$\mathcal{I}(y_\ell,1,\nu) \simeq y(1-y^2)^\nu e^{2\varpi_{\rm E} y}\big|_{y=y_\ell}(1-y_\ell) \simeq \frac{e^{2\varpi_{\rm E}}}{2}\left(\frac{\ell}{\varpi_0}\right)^{2(\nu+1)}. \tag{9.128}$$

For the integrals of the type $\mathcal{I}(0,y_\ell,\nu)$ we distinguish between three cases:

Case a: $\nu > -1$. This case corresponds to a spectral index $n > 1/2$ in the first integral in Eq. (9.125). We write $\mathcal{I}(0,y_\ell,\nu) = \mathcal{I}(0,1,\nu) - \mathcal{I}(y_\ell,1,\nu)$. The solution of the latter is given by Eq. (9.128), whereas the former can be solved in terms of modified Bessel and Struve functions [70]

$$\mathcal{I}(0,1,\nu) = \frac{1}{2(\nu+1)} + \frac{\sqrt{\pi}}{2}\varpi_{\rm E}^{-1/2-\nu}\Gamma(\nu+1)\left[{\rm I}_{\nu+3/2}(2\varpi_{\rm E}) + {\rm L}_{\nu+3/2}(2\varpi_{\rm E})\right]. \tag{9.129}$$

Since our derivation assumes $\varpi_{\rm E} > \ell$, the large argument limit applies and we have

$${\rm I}_{\nu+3/2}(2\varpi_{\rm E}) + {\rm L}_{\nu+3/2}(2\varpi_{\rm E}) \simeq \frac{e^{2\varpi_{\rm E}}}{\sqrt{\pi\varpi_{\rm E}}}, \tag{9.130}$$

independently of the index ν.

Case b: $\nu = -1$. Since the above expressions, Eq. (9.129), diverge for $\nu = -1$, we approximate the integral by

$$\mathcal{I}(0,y_\ell,\nu) \simeq e^{2\varpi_{\rm E}}\int_0^{y_\ell} y(1-y^2)^{-1}{\rm d}y = -e^{2\varpi_{\rm E}}\ln\left(\frac{\ell}{\varpi_0}\right). \tag{9.131}$$

We have checked that the numerical solution of $\mathcal{I}(0,y_\ell,\nu)$ agrees well with the approximation, provided y_ℓ is close to 1.

Case c: $\nu < -1$. We use the same simplification as in Eq. (9.131), and now the integral yields

$$\mathcal{I}(0,y_\ell,\nu) \simeq e^{2\varpi_{\rm E}}\int_0^{y_\ell} y(1-y^2)^\nu{\rm d}y = -\frac{e^{2\varpi_{\rm E}}}{2(\nu+1)}\left(\frac{\ell}{\varpi_0}\right)^{2(\nu+1)}. \tag{9.132}$$

For the particular value $n = -3/2$, Eq. (9.127) contains terms $\mathcal{I}(0, y_\ell, -3)$ and $\mathcal{I}(0, y_\ell, -2)$ which can be evaluated according to (9.132), and a term $\mathcal{I}(0, y_\ell, -1)$ for which we use (9.131). The remaining three integrals over the interval $[y_\ell, 1]$ are evaluated by (9.128). The result is

$$C_\ell^{(1)} \simeq \frac{\mathrm{e}^{\varpi_0/z_\mathrm{E}}}{4\pi\ell^4}\left[1 - \frac{\ell^2}{6z_\mathrm{E}} - \left(\frac{\ell^2}{12z_\mathrm{E}}\right)^2 \ln\left(\frac{\ell}{\varpi_0}\right) + \frac{\mathrm{e}^{2\beta\ell}}{2}\left(1 - \frac{\ell^2}{24z_\mathrm{E}}\right)^2\right]. \tag{9.133}$$

The parameter β is a constant of order unity, within in our approximation it is $\beta = 1 - \ln 2 \sim 0.3$.

The second term $C_\ell^{(2)}$ in Eq. (9.123) reads

$$\begin{aligned} C_\ell^{(2)} \simeq & \frac{1}{3\ell^{7/2}\varpi_0^{3/2s}} \int_0^1 \mathrm{e}^{(\varpi_0+\varpi_\mathrm{E})\sqrt{1-\bar{k}^2}}\bar{k}^{2n+\ell-1/2} \\ & \times \sqrt{1-\bar{k}^2}\left(1 - \frac{\bar{k}^2}{\bar{k}_\mathrm{E}^2}\right) \mathrm{J}_{\ell+1/2}(\bar{k}\varpi_0)\mathrm{d}\bar{k}, \end{aligned} \tag{9.134}$$

where only the zero order terms in $\varpi_\mathrm{E}/\varpi_0$ has been kept. Using the limiting forms for the Bessel function for arguments smaller and larger than the transition value $\bar{k}_\ell$, yields

$$\begin{aligned} C_\ell^{(2)} \simeq & \frac{1}{3\ell^{7/2}\varpi_0^{3/2}\Gamma(\ell+3/2)} \int_0^{\bar{k}_\ell} \mathrm{e}^{(\varpi_0+\varpi_\mathrm{E})\sqrt{1-\bar{k}^2}}\bar{k}^{2n+\ell-1/2} \\ & \times \sqrt{1-\bar{k}^2}\left(1 - \frac{\bar{k}^2}{\bar{k}_\mathrm{E}^2}\right)\left(\frac{\bar{k}\varpi_0}{2}\right)^{\ell+1/2}\mathrm{d}\bar{k} \\ & + \frac{2^{1/2}}{3\pi^{1/2}\ell^{7/2}\varpi_0} \int_{\bar{k}_\ell}^1 \mathrm{e}^{(\varpi_0+\varpi_\mathrm{E})\sqrt{1-\bar{k}^2}}\bar{k}^{2n+\ell-1} \\ & \times \sqrt{1-\bar{k}^2}\left(1 - \frac{\bar{k}^2}{\bar{k}_\mathrm{E}^2}\right)\sin\left(\bar{k}\varpi_0 - \frac{\pi}{2}\ell\right)\mathrm{d}\bar{k}. \end{aligned} \tag{9.135}$$

For consistency with the derivation of $C_\ell^{(1)}$, we have assumed that the main contribution comes from the first integral, while the second one is small due to the oscillating integrand. Since $\bar{k}_\ell \ll 1$, we can use again the mean value formula to evaluate the first integral, and by the Stirling formula for $\Gamma(\ell+3/2)$, Eq. (9.135) becomes

$$C_\ell^{(2)} \simeq \frac{\mathrm{e}^{\varpi_0}}{12\pi^{1/2}\ell^{11/2}}\left(\frac{\ell}{\varpi_0}\right)^{2n+\ell+2}\mathrm{e}^{\beta\ell}\left(1 - \frac{\ell^2}{24z_\mathrm{E}}\right). \tag{9.136}$$

Since $\ell < \varpi_0$, the spectrum is damped at large ℓ, while the other terms can lead to the appearance of a bump, depending on the value of ϖ_0 and n.

The last terms $C_\ell^{(3)}$ reads

$$C_\ell^{(3)} = \frac{1}{36\ell^7}\int_0^1 \mathrm{e}^{2\varpi_0\sqrt{1-\bar{k}^2}}\bar{k}^{2n+2\ell}\left(1 - \bar{k}^2\right)\mathrm{d}\bar{k}. \tag{9.137}$$

Splitting this expression in two terms over $1-\bar{k}^2$, and using the substitution $y^2 = 1-\bar{k}^2$ yields

$$C_\ell^{(3)} = \frac{1}{36\ell^7}\left[\mathcal{I}\left(0,1,n+\ell-\frac{1}{2}\right) - \mathcal{I}\left(0,1,n+\ell+\frac{1}{2}\right)\right], \tag{9.138}$$

where $\mathcal{I}$ is given by Eq. (9.126) with $\varpi_{\rm E} \to \varpi_0$. As before, these two integrals can be expressed in terms of modified Bessel and Struve functions [70]. From Eq. (9.129), taking their limiting forms at large argument, and expanding the Γ function by means of the Stirling formula gives

$$C_\ell^{(3)} \simeq \frac{1}{72\ell^7\,(n+\ell+3/2)\,(n+\ell+1/2)} + \sqrt{\frac{\pi}{2}}\frac{{\rm e}^{2\varpi_0}}{\ell^{15/2}}\frac{{\rm e}^{-\ell}}{36}\left(\frac{\ell}{\varpi_0}\right)^{n+\ell+1/2}. \tag{9.139}$$

Clearly, $C_\ell^{(3)}$ dominates over the others since it involves $\exp(2\varpi_0)$ while $C_\ell^{(2)}$ and $C_\ell^{(1)}$ appear only with fractional power of this factor, namely $\exp(\varpi_0)$ and $\exp(\varpi_0/z_{\rm E})$. This is due to the fact that we are concerned with incessantly growing perturbations leading to the predominance of the integrated Sachs-Wolfe effect.

Inserting Eqs. (9.133), (9.136) and (9.139) for the particular value $n=-3/2$ into Eq. (9.124) gives the final CMB angular power spectrum

$$\frac{\ell(\ell+1)}{2\pi}C_\ell \simeq \frac{2}{\pi}A_0^2\left\{\frac{{\rm e}^{\varpi_0/z_{\rm E}}}{4\pi}\left[1-\frac{\ell^2}{6z_{\rm E}} - \left(\frac{\ell^2}{12z_{\rm E}}\right)^2\ln\left(\frac{\ell}{\varpi_0}\right) + \frac{{\rm e}^{2\beta\ell}}{2}\left(1-\frac{\ell^2}{24z_{\rm E}}\right)^2\right]\right.$$
$$\left. + \frac{{\rm e}^{\varpi_0}}{12\pi^{1/2}\ell^{3/2}}\left(\frac{\ell}{\varpi_0}\right)^{\ell-1}{\rm e}^{\beta\ell}\left(1-\frac{\ell^2}{24z_{\rm E}}\right) + \sqrt{\frac{\pi}{2}}\frac{{\rm e}^{2\varpi_0}}{\ell^{7/2}}\frac{{\rm e}^{-\ell}}{36}\left(\frac{\ell}{\varpi_0}\right)^{\ell-1}\right\}. \tag{9.140}$$

Part III

BRANE GAS COSMOLOGY

Chapter 10

The cosmology of string gases

In the last part of this thesis, we are going to address two main questions: first, how can one avoid the initial singularity in the standard cosmology, and second, why does our universe have apparently *three* spatial dimensions? As different as they are, both questions can possibly be resolved in a single framework called 'string gas' or 'brane gas' cosmology. This proposal is quite different than the brane world models we have encountered so far.

Roughly speaking, brane world models fall in two categories: models in which one or two parallel 3-branes are at orbifold fixed points of a compact or non compact extra-dimension, and models in which a 3-brane is moving through an anti-de Sitter space-time. Both (in the end equivalent) scenarii are based on a compactification of 11-dimensional M theory down to a five-dimensional space-time (see paragraph 2.5.3).

In contrast, in brane gas cosmology all nine spatial dimensions predicted by superstring theory are compact from the start. The matter source is a variety of p-branes and strings, and the term 'gas' is used because they are homogeneously distributed in space and have no particular orientation with respect to each other. In contrast to brane world models, we do not live on one of these branes, but simply somewhere in the bulk.

In the next two sections, 10.1 and 10.2, we closely follow the original work of Brandenberger and Vafa [29], where only fundamental strings were considered. An extension including p-branes was made in Ref. [5]. The aim of the article 'On T-duality in branes gas cosmology' in the next chapter is to show that the arguments for a non singular cosmological evolution given in Ref. [29] carry over when branes are also included.

10.1 Avoiding the initial singularity

10.1.1 T-duality

Crucial in string and brane gas cosmology is the concept of T-duality introduced in Sec. 3.4. Let us consider closed fundamental strings moving in a compact space which, for simplicity, is taken to be a 9-dimensional torus. The strings form a gas in the sense mentioned above. The possible string states are oscillatory modes, corresponding to the vibration of the string, momentum modes, corresponding to the center of mass motion, and winding modes, describing the winding of a string around a compact direction. Each excited state contributes to the energy of a string which is (for a derivation see Sec. 3.4)

$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{\omega R}{\alpha'}\right)^2 + \text{ oscillators.} \tag{10.1}$$

Here, R denotes the radius of a compact direction and plays the role of a dimensionful scale factor. The square root of the constant α' is the fundamental string length, and the integers n and ω are the excitation numbers of the momentum modes and the winding modes. The latter is counting how many times a closed string winds around a compact direction. Expression (10.1) is invariant under the transformation

$$R \to \frac{\alpha'}{R}, \quad n \to \omega, \quad \omega \to n, \tag{10.2}$$

i.e. under the exchange of large and small lengths, as well as of the simultaneous exchange of momentum and winding modes. This symmetry is called T-duality, and there are strong arguments that it holds not only in this particular example, but that it is an intrinsic and fundamental symmetry of the whole string theory. All physical processes should be equivalent, whether they are looked at with the 'R-eye' or the '$1/R$-eye'. This is possible because the notion of distance is a derived concept. It can be defined either as the Fourier transform of momentum states

$$|x\rangle = \sum_p \mathrm{e}^{ix\cdot p}\,|p\rangle\,, \quad \text{with} \quad |x\rangle = |x + 2\pi R\rangle\,, \tag{10.3}$$

or as the Fourier transform of winding states

$$|x'\rangle = \sum_\omega \mathrm{e}^{ix'\cdot\omega}\,|\omega\rangle\,, \quad \text{with} \quad |x'\rangle = |x' + 2\pi\alpha'/R\rangle\,. \tag{10.4}$$

Note that in R-space the winding modes are not localized. In the $1/R$ space, however, they are localized, since they correspond to momentum states in R-space. Neither of the two definitions (10.3) and (10.4) is more fundamental. For example, there is no experiment which could decide whether we live in a universe which is 10^{10} or 10^{-10} lightyears wide.

Effectively, there exists a minimal length $R = \alpha'^{1/2}$, in the sense that the range $R < \alpha'^{1/2}$ can be equally well described using the R'-picture where $R' > \alpha'^{1/2}$. Ultimately, this is due to the extended nature of strings, and it is the key to formulate a singularity-free theory. We shall see below how this affects the behavior of the temperature as $R \to 0$.

The Friedmann equation in the standard 4-dimensional cosmology, $\dot{a}/a \sim 1/a^4$ or $1/a^3$, where $a \propto R$, is not invariant under T-duality. This is a hint that Einstein's equations break down at least at the self-dual radius $\alpha'^{1/2}$, and that the standard cosmology is valid only for $R \gg \alpha'^{1/2}$. A reason why the usual Friedmann equations lacks T-duality symmetry is that usually space-time is considered to be non compact, and therefore the winding modes are absent. To restore T-duality symmetry in a cosmological equation, one would have to include a graviton associated with the winding modes.

10.1.2 String thermodynamics

As one approaches the initial big bang singularity (going backwards in time) the spatial sections shrink to zero size, and the temperature rises up to infinity. The singularity theorems (see paragraph 1.3.3) state that this is unavoidable for matter satisfying the strong energy condition[1] (under the assumption that Einstein's equations are valid). Surprisingly, in string thermodynamics this temperature singularity does not show up. To see this, consider the canonical[2] partition function for a single string

$$Z = \int_0^\infty \mathrm{d}E\,\rho(E)\mathrm{e}^{-E/T}, \tag{10.5}$$

where

$$\rho(E) = \mathrm{e}^{4\pi\alpha'^{1/2}E} \tag{10.6}$$

is the number density (degeneracy) of string states of energy E (see Ref. [124]). Note that, due to the exponential growth, the degeneracy becomes rapidly enormous. The integral in Eq. (10.5) is finite provided that the temperature T is smaller than

$$T_H \equiv \frac{1}{4\pi\alpha'^{1/2}} \tag{10.7}$$

which is called Hagedorn temperature[3]. This suggests that there exists a maximum attainable temperature: for $T > T_H$, the partition function diverges, and therefore the mean energy

$$E_{\text{mean}} = -T^2 \frac{\partial}{\partial T} \ln Z \tag{10.8}$$

would be infinite. This is in contrast to the partition function for point particles.

How does the temperature of the string gas behave as a function of the (dimensionful) scale factor R as we go to smaller and smaller radii R? To answer this question we make two assumptions: firstly, that the back-reaction of the string gas on the space-time geometry can be neglected. This is justified if the string coupling constant g_s is much smaller than one. And secondly, that the evolution of the universe is adiabatic, *i.e.* the entropy remains constant. This second assumption allows us to find $T(R)$ without knowing the particular dynamics of gravity. We can already guess the qualitative shape of the $T(R)$ curve: at large R, the $T \sim R^{-1}$ behavior in the standard cosmology should be recovered. As one goes to smaller radii, the curve will flatten out at a temperature close to T_H, and at even smaller radii the temperature drops again in order to satisfy the T-duality symmetry

$$T(R) = T\left(\frac{\alpha'}{R}\right). \tag{10.9}$$

[1]For a perfect fluid this is $\rho + 3P \geq 0$.

[2]We remind the reader that a canonical ensemble is a very large number of copies of the original system that all have the same temperature. This is realized for example by bringing them into thermal contact with a thermostat of temperature T. However, the system is still exchanging heat with the thermostat, and therefore its energy is fluctuating. The probability of finding a state with energy E is encoded in the exponential factor in Eq. (10.5). Notice that T is an external parameter, whereas E is an inner variable of the system.

[3]The original work of Hagedorn on thermodynamics of strong interactions can be found under Ref. [74]

The maximum temperature is reached at the self-dual radius $R = \alpha^{'1/2}$. A careful computation shows that this corresponds not quite to the Hagedorn temperature, but to $T = T_H - c/S^2$, where c is some constant, and S is the entropy of the string gas. The precise shape of the curve has been calculated numerically in Ref. [29] [4].

A physical interpretation is the following: at large R the winding modes are irrelevant, and all energy is stored in momentum modes (see Eq. (10.1)). Since these correspond to a center of mass motion, one expects the same behavior of temperature as in a gas of pointlike particles, namely an increase in T as R decreases. On the other hand, as $R \to 0$, it becomes energetically favorable to excite winding modes instead of momentum modes, and an energy transfer to the latter takes place. When R is very close to zero, the spacing between winding states (given by R) becomes zero, such that they are in the ground state corresponding to zero temperature. In conclusion, the temperature must be zero both for $R \to 0$ and $R \to \infty$, with a maximum in between.

From the point of view of T-duality, the absence of a temperature singularity can be understood by noticing that for an observer with the R'-eye, the collapse $R \to 0$ means an expansion $R' \to \infty$, and so it is clear that the temperature has to drop to zero. These arguments show that the initial singularity can be avoided in a cosmological scenario with strings as a matter source.

The authors of Ref. [29] also pointed out that close to the Hagedorn temperature the energy fluctuations become large, and one therefore has to resort to the microcanonical ensemble[5]. In particular, they showed that at sufficiently large energies the specific heat becomes negative, unless all spatial directions are compact. That is an important reason why in string gas cosmology, and later also in brane gas cosmology, only compact spaces are considered.

10.2 Why is space three-dimensional?

We have seen that in string gas cosmology all nine spatial dimensions are compact from the beginning. In brane world models instead, one starts with a non compact space and then compactifies a number of spatial dimensions. A serious shortcoming of this procedure is that the effective 4-dimensional physics will depend on the geometry of the compact space. For example, we have seen that Newton's constant depends on the volume of the extra-dimensions (see Eq. (2.68)). More generally, each compactification involves a large number of so-called modulus fields describing the shape and size of the compact space. In the simplest case of toroidal compactification, such a modulus field is *e.g.* the radius R. Unfortunately, it is to a large extent arbitrary on which particular space (a torus, a Calabi-Yau manifold or others) the compactification must be carried out. Certain restrictions come from the requirement of obtaining the standard model gauge groups

[4] Strictly speaking, all this applies only for type II superstrings. The situation is different for heterotic strings, but it can be argued that they lead to a similar curve $T(R)$.

[5] In the microcanonical ensemble all systems have an energy contained in an interval $[E, E + \delta E]$, which is by definition not fluctuating as in the canonical ensemble.

or a certain number of supersymmetries. Still, the parameter space is huge, and therefore string theory looses virtually all of its predictive power. This is one of the main problems today in connecting string theory to low energy physics. Note that string theory itself has only one free parameter, namely α'.

Therefore, to avoid the moduli problem and to have a well defined thermodynamical description, string gas cosmology postulates that all nine spatial dimensions are compact from the start. The initial conditions are that all directions are small and of equal (string scale) size[6], and that the strings form a hot and dense gas.

Given these 'natural' initial conditions one would like to explain why finally three spatial dimensions have become much larger that the others. Such a dynamical 'decompactification' mechanism has been proposed in Ref. [29] for the case that the compact manifold is a 9-torus. Let us assume that initially all nine directions are of string scale size and start to expand isotropically. The winding modes wrapping around all 1-cycles would like to slow down the expansion since their energy is proportional to the radius R of a cycle. However, their number decreases as the size of the torus increases in order to minimize the energy. Hence the expansion can continue. This is true provided that the winding modes are in thermal equilibrium with other modes, string loops, and radiation. As soon as they fall out of thermal equilibrium, a large number of winding modes remain, and this will stop the expansion. The process that maintains thermal equilibrium is of the form

$$\omega + \bar{\omega} \leftrightarrow \text{unwound states}, \tag{10.10}$$

where $\bar{\omega}$ stands for a string with opposite winding number, and unwound states are, for instance, string loops that have split off or radiation. The annihilation (10.10) between a winding and an anti-winding state happens by the intercommutation process shown in Fig. 10.1. Here we assume that, as the cycles grow larger, the strings behave as classical objects with a thickness given by the string length.

The crucial point is now that, in order for the process (10.10) to happen, the two world-sheets of the strings have to intersect. In a 9-dimensional space, strings will generically miss each other, so the winding modes fall out of equilibrium and stop the expansion of the torus. In a 3-dimensional subspace, however, the winding modes will generically meet, because their world-sheets fill all of spacetime: $(1 + 1) + (1 + 1) = 3 + 1$. Thus, in a 3-dimensional subspace, thermal equilibrium can be maintained due to the process (10.10), and the expansion can go on, while the other dimensions stay small. This is called the Brandenberger-Vafa mechanism to dynamically explain, why we see three large spatial dimensions today. Notice that, according to this scenario, the three large dimensions are still compact. This is not in contradiction with observations as long as their radius is much larger than the Hubble radius.

[6]This is the third crucial assumption, together with the smallness of the string coupling constant, and the adiabaticity of the evolution.

Figure 10.1: A toroidal topology can be represented as a cube, where opposite faces are identified. The left panel shows two closed strings winding around different compact directions. These strings will generically intersect each other if they are propagating in 3-dimensional space. After the intersection, the strings regroup as displayed in the right panel. Effectively, this process corresponds to an unwinding from the compact direction.

These rather qualitative arguments are supported by numerical simulations [137] and by the equations of motion of string cosmology which govern the evolution of the background: in Ref. [153] it was demonstrated that the winding modes, if not annihilated, indeed stop the expansion because of their negative pressure.

There is a number of points that remain to be clarified. First, if the expansion rate $\dot{R}/R$ of the torus if too slow, all processes will equilibrate, and in particular the winding modes do not fall out of equilibrium anymore. One should therefore quantitatively compare the time scales of the two processes. Second, what happens if there is a slight asymmetry between the number of winding and anti-winding modes? (Recall the asymmetry between matter and anti-matter in the early universe, which is crucial for our existence.) Third, one should check whether the classical treatment of strings is justified. Forth, it is not clear why the number of large spatial dimensions is not less than three: after all, it is more efficient to maintain thermal equilibrium in one ore two dimensions. Fifth, the scenario does not take into account long distance interactions. According to the classical dimension counting argument, point particles would always fail to meet each other and to interact, unless they live on a line. And finally, in this discussion we have not yet taken into account p-branes, which are important objects in string theory. The role of p-branes within the BV mechanism has been studied in Ref. [5], and the question of avoiding the initial singularity in a theory with p-branes will be investigated in the article in the next chapter.

Despite these shortcomings, a dynamical decompactification mechanism seems more 'natural' than starting from an non compact space-time and compactifying by hand six dimensions. In particular, those scenarios do not explain why one ends up with three large dimensions.

The dynamical decompactification process takes place at very early times and high energies, and so experimental confirmation is lacking. Perhaps, it is at least possible to verify whether we live in a compact universe or not by cosmic crystallography, see *e.g.* Ref. [106].

10.3 Equation of state of a brane gas

We now calculate the equation of state of a gas consisting of p-branes in a D-dimensional space-time. This section gives the detailed version of the calculation presented in the article 'On T-duality in brane gas cosmology' in the next Chapter. The equation of state is important for cosmological issues such as brane gas inflation.

We start our calculation from the D-dimensional energy-momentum tensor describing a single p-brane whose location is encoded in the embedding functions $X^K(\sigma)$,

$$T^{MN}(x^K) = -\tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma))\sqrt{-g}\, g^{\mu\nu} X^M_{,\mu} X^N_{,\nu}. \tag{10.11}$$

For simplicity, the ensemble of internal coordinates σ^μ is abbreviated as σ. Here, $\tau_p = T_p/g_s$ is the physical brane tension, $g_{\mu\nu}$ is the induced metric on the brane, and $g = \det(g_{\mu\nu})$. Since we are interested in the energy density and the pressure of a gas of arbitrarily oriented p-branes, we do not fix a particular embedding and leave the functions X^K unspecified. Instead, we use the freedom to reparameterize the $p+1$ intrinsic world-sheet coordinates σ^μ to choose a gauge in which $g_{00} = -\sqrt{-g}, g_{01} = 0, \cdots, g_{0p} = 0$. The induced metric is then

$$(g_{\mu\nu}) = \begin{pmatrix} -\sqrt{-g} & 0 & \cdots & 0 \\ 0 & g_{11} & \cdots & g_{1p} \\ \vdots & \vdots & & \vdots \\ 0 & g_{p1} & \cdots & g_{pp} \end{pmatrix}, \tag{10.12}$$

and its inverse is given by

$$(g^{\mu\nu}) = \begin{pmatrix} -1/\sqrt{-g} & 0 & \cdots & 0 \\ 0 & g^{11} & \cdots & g^{1p} \\ \vdots & \vdots & & \vdots \\ 0 & g^{p1} & \cdots & g^{pp} \end{pmatrix}. \tag{10.13}$$

Notice that the spatial $p \times p$ submatrices are also inverses to each other. Furthermore, we make a gauge choice for the time coordinate σ^0

$$X^0(\sigma) = \sigma^0. \tag{10.14}$$

Then the energy density of a p-brane is

$$\begin{aligned} T^{00}(x^K) &= -\tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma))\sqrt{-g}\, g^{\mu\nu} X^0_{,\mu} X^0_{,\nu} \\ &= -\tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma))\sqrt{-g} \\ &\times \left[g^{00} X^0_{,0} X^0_{,0} + g^{11} X^0_{,1} X^0_{,1} + g^{12} X^0_{,1} X^0_{,2} + \cdots\right], \end{aligned} \tag{10.15}$$

but because of the gauge choice (10.14), all terms with spatial derivatives are zero. Inserting the expression for g^{00} one obtains

$$\begin{aligned} T^{00}(x^K) &= -\tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma))\sqrt{-g}(-1/\sqrt{-g}) \\ &= +\tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma)). \end{aligned} \tag{10.16}$$

The delta function is split according to

$$\delta^{(D)}(x^K - X^K(\sigma)) = \delta(t - \sigma^0)\delta^{(d)}(x^n - X^n(\sigma^0, \sigma^i)), \tag{10.17}$$

where $\sigma^\mu = (\sigma^0, \sigma^i), x^K = (t, x^n)$, $d = D - 1$. Finally, one integrates out time to obtain the energy density of a p-brane in d spatial dimensions:

$$\rho_p = T^{00}(t, x^n) = +\tau_p \int \mathrm{d}^p\sigma \delta^{(d)}(x^n - X^n(t, \sigma^i)). \tag{10.18}$$

Next, we calculate the (spatial) trace of the energy-momentum tensor, noting that the background is Minkowski with signature $(-, +, \cdots, +)$,

$$\begin{aligned} T^m_m &= T^1_1 + \cdots + T^d_d = T^{11} + \cdots + T^{dd} \\ &= -\tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma))\sqrt{-g} \\ &\times \left[g^{\mu\nu} X^1_{,\mu} X^1_{,\nu} + \cdots + g^{\mu\nu} X^d_{,\mu} X^d_{,\nu}\right]. \end{aligned} \tag{10.19}$$

Collecting the metric coefficients yields

$$\begin{aligned} T^m_m(x^K) = -\tau_p \int & \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma))\sqrt{-g} \\ \times \Big[& g^{00}(X^1_{,0} X^1_{,0} + \cdots + X^d_{,0} X^d_{,0}) \\ &+ g^{11}(X^1_{,1} X^1_{,1} + \cdots + X^d_{,1} X^d_{,1}) + \cdots \\ &+ g^{pp}(X^1_{,p} X^1_{,p} + \cdots + X^d_{,p} X^d_{,p}) \\ &+ 2g^{12}(X^1_{,1} X^1_{,2} + \cdots + X^d_{,1} X^d_{,2}) + \cdots \\ &+ 2g^{1p}(X^1_{,1} X^1_{,p} + \cdots + X^d_{,1} X^d_{,p}) \\ &+ 2g^{23}(X^1_{,2} X^1_{,3} + \cdots + X^d_{,2} X^d_{,3}) + \cdots \\ &+ 2g^{2p}(X^1_{,2} X^1_{,p} + \cdots + X^d_{,2} X^d_{,p}) + \cdots \\ &+ 2g^{p-1,p}(X^1_{,p-1} X^1_{,p} + \cdots + X^d_{,p-1} X^d_{,p})\Big]. \end{aligned} \tag{10.20}$$

The idea is now to re-express the derivatives of the embedding functions in terms of $g_{\mu\nu}$. For instance

$$\begin{aligned} g_{12} &= -X^0_{,1}X^0_{,2} + X^1_{,1}X^1_{,2} + \cdots + X^d_{,1}X^d_{,2} \\ &= X^1_{,1}X^1_{,2} + \cdots + X^d_{,1}X^d_{,2}, \end{aligned} \tag{10.21}$$

where in the second line we have used the fact that the first term vanishes according to gauge (10.14). With $X^m = (X^1, \cdots, X^d)$ one finds

$$\begin{aligned} T^m_m(x^K) = &-\tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma))\sqrt{-g} \\ &\times \Big[g^{00}\dot{X}^m\dot{X}_m + g^{11}g_{11} + \cdots + g^{pp}g_{pp} \\ &+ 2g^{12}g_{12} + \cdots + 2g^{1p}g_{1p} \\ &+ 2g^{23}g_{23} + \cdots + 2g^{2p}g_{2p} + \cdots \\ &+ 2g^{p-1,p}g_{p-1,p}\Big]. \end{aligned} \tag{10.22}$$

The dot denotes a derivative with respect to σ^0. Next, one uses the fact that the spatial submatrices g^{ij} and g_{ij} are also inverses

$$\begin{aligned} 1 &= g^{11}g_{11} + g^{12}g_{21} + \cdots + g^{1p}g_{p1} \\ &= g^{11}g_{11} + g^{12}g_{12} + \cdots + g^{1p}g_{1p}, \end{aligned} \tag{10.23}$$

such that T^m_m can be simply written

$$T^m_m(x^K) = -\tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma))\sqrt{-g}\left[(-1/\sqrt{-g})\dot{X}^m\dot{X}_m + p\right]. \tag{10.24}$$

To eliminate the remaining $\sqrt{-g}$, one uses the gauge choice for g_{00}:

$$-\sqrt{-g} = g_{00} = -X^0_{,0}X^0_{,0} + X^1_{,0}X^1_{,0} + \cdots + X^d_{,0}X^d_{,0} = -1 + \dot{X}^m\dot{X}_m. \tag{10.25}$$

Substituting this in Eq. (10.24) and integrating out time, the final result for the (spatial) trace of the energy-momentum tensor is

$$T^m_m(t, x^n) = +\tau_p \int \mathrm{d}^p\sigma \delta^{(d)}(x^n - X^n(t, \sigma^i))\left[(p+1)\dot{X}^m\dot{X}_m - p\right]. \tag{10.26}$$

The quantity $\dot{X}^m\dot{X}_m$ is the squared velocity of a point on the brane parameterized by (t, σ^i). To take into account all points and all embeddings, one has to average over the spatial coordinates σ^i. This leads to a 'brane gas averaged' velocity $v^2(t) = \langle\dot{X}^m\dot{X}_m\rangle$ which depends on time only, and hence can be taken out of the integral:

$$\langle T^m_m(t, \vec{x})\rangle = \left[(p+1)v^2 - p\right]\tau_p \int \mathrm{d}^p\sigma \delta^{(d)}(x^n - X^n(t, \sigma^i)). \tag{10.27}$$

Comparing Eq. (10.27) with the expression for the energy density (10.18), one finds the equation of state

$$\mathcal{P}_p = \frac{1}{d}\langle T^m_m \rangle = \left[\frac{p+1}{d}v^2 - \frac{p}{d}\right]\rho_p, \tag{10.28}$$

where $\mathcal{P}_p$ is the pressure of the p-brane gas. This pressure is within the range

$$-\frac{p}{d}\rho_p \leq \mathcal{P}_p \leq \frac{1}{d}\rho_p, \tag{10.29}$$

where the lower bound is assumed in the non relativistic limit, $v \to 0$, and the upper bound in the relativistic limit, $v \to 1$.

To illustrate this formula, take for example a domain wall in three spatial dimensions ($p = 2, d = 3$). Then the equation of state (10.28) gives

$$\mathcal{P}_2 = \left[v^2 - \frac{2}{3}\right]\rho_2, \tag{10.30}$$

which in the limit $v \to 0$ gives $\mathcal{P}_2 = -\frac{2}{3}\rho_2$ as expected, and for $v \to 1$ gives $\mathcal{P}_2 = \frac{1}{3}\rho_2$.

It is amusing to note that the case $p = d$ in the non relativistic limit leads to an equation of state $\mathcal{P} = -\rho$. Hence a cosmological constant can be interpreted as a space-filling brane.

Chapter 11

On T-duality in brane gas cosmology (article)

This chapter consists of the article 'On T-duality in brane gas cosmology', published in JCAP 06 (2003), see Ref. [20].

It is also available under http://lanl.arXiv.org/abs/hep-th/0208188.

Like in the previous chapter, the pressure of a p-brane gas will be denoted by $\mathcal{P}_p$, because the letter P is here used for the p-brane momentum.

On T-duality in brane gas cosmology

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In the context of homogeneous and isotropic superstring cosmology, the T-duality symmetry of string theory has been used to argue that for a background space-time described by dilaton gravity with strings as matter sources, the cosmological evolution of the universe will be non singular. In this letter we discuss how T-duality extends to brane gas cosmology, an approximation in which the background space-time is again described by dilaton gravity with a gas of branes as a matter source. We conclude that the arguments for non singular cosmological evolution remain valid.

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11.1 Introduction

In [29] it was suggested that due to a new string theory-specific symmetry called T-duality, string theory has the potential to resolve the initial singularity problem of standard big bang cosmology, a singularity which also plagues scalar field-driven inflationary cosmology [25, 24].

The framework of [29] was based on an approximation in which the mathematical background space-time is described by the equations of dilaton gravity (see [153, 157]), with the matter source consisting of a gas of strings. The background spatial sections were assumed to be toroidal such as to admit one-cycles. Thus, the degrees of freedom of the string gas consist of winding modes in addition to the momentum modes and the oscillatory modes. Then, both momentum and winding numbers take on discrete values, and the energy spectrum of the theory is invariant under inversion of the radii of the torus, *i.e.* $R \to \alpha'/R$, where $\alpha'^{1/2}$ is the string length l_s. The mass of a state with momentum and winding numbers n and ω, respectively, in a compact space of radius R, is the same as that of a state with momentum and winding numbers ω and n, respectively, in the space of radius α'/R. This symmetry was used to argue [29] that as the radii of the torus decrease to very low values, no physical singularities will occur. Firstly, under the assumption of thermal equilibrium, the temperature of a string gas at radius R will be equal to the temperature of the string gas at radius α'/R. Secondly, any process computed for strings on a space with radius R, is identical to a dual process computed for strings on a space with radius α'/R. Therefore there exists a 'minimal' radius $\alpha'^{1/2}$ in the sense that physics on length scales below this radius can equally well be described by physics on length scales larger than that.

Since the work of [29] our knowledge of string theory has evolved in important ways. In particular, it has been realized [126] that string theory must contain degrees of freedom other than the perturbative string degrees of freedom used in [29]. These new degrees of freedom are Dp-branes of various dimensionalities (depending on which string theory one is considering). Since the T-duality symmetry was used in an essential way (see *e.g.* [125]) to arrive at the existence of Dp-branes (p-branes for short in the following), it is clear that T-duality symmetry will extend to a cosmological scenario including p-branes. However, since a T-duality transformation changes the dimensionality of branes, it is useful to explicitly verify that the arguments of [29] for a non singular cosmological evolution carry over when the gas of perturbative string modes is generalized to a gas of branes. A model for superstring cosmology in which the background space-time is described (as in [153, 157]) by dilaton gravity, and the matter source is a gas of branes, has recently been studied under the name of "brane gas cosmology" [5, 27] (see also [57, 58] for extensions to backgrounds which are not toroidal, and [161] for an extension to an anisotropic background).

In this letter, we establish the explicit action of T-duality in the context of brane gas cosmology on a toroidal background. For a solution of the background geometry appropriate for cosmological considerations in which the radii of the torus are decreasing from large to small values as we go back in time, we must consider T-dualizing in all spatial dimensions. We demonstrate that the mass spectrum of branes remains invariant under this action. Thus, if the background dynamics are adiabatic, then the temperature of the brane gas will be invariant under the change $R \to \alpha'/R$, *i.e.*

$$T(R) = T\left(\frac{\alpha'}{R}\right), \tag{11.1}$$

thus demonstrating that superstring cosmology can avoid the temperature singularity problem of standard and inflationary cosmology. In the appendix we make further remarks on why we expect brane gas cosmology to be non singular. Another crucial assumption is that the string coupling constant g_s is small (compared to one) such that back-reactions of the string and brane gas on the curvature of space-time can be ignored. Similarly, our results can be used to show that the arguments for the existence of a minimal physical length given in [29] extend to brane gas cosmology.

The outline of this letter is as follows. In the following section we give a brief review of brane gas cosmology. In section 11.3 we derive the energy, the momentum, and the pressure for p-branes. Next, we review the action of T-duality on winding states of p-branes. The main section of this letter is section 11.5 in which we show that the mass spectrum of a p-brane gas of superstring theory is invariant under T-duality. Section 11.6 contains a discussion of some implications of the result, and conclusions.

11.2 Review of brane gas cosmology

As already mentioned in the introduction, the framework of brane gas cosmology consists of a homogeneous and isotropic background of dilaton gravity coupled to a gas of p-branes as a matter source. We are living in the bulk[1].

The initial conditions in the early universe are 'conservative' and 'democratic'; conservative in the sense that they are close to the initial conditions assumed to hold in standard big bang cosmology (*i.e.* a hot dense gas of matter), democratic in the sense that all 9 spatial dimensions of critical string theory are considered on an equal basis[2]. Thus, matter is taken to be a gas of p-branes of all allowed values of p in thermal equilibrium. In particular, all modes of the branes are excited, including the winding modes.

The background space-time is taken to be $\mathbb{R} \times T^9$ where T^9 denotes a nine-torus. The key feature of T^9 which is used in the analysis is the fact that it admits one-cycles which makes it possible for closed strings to have conserved winding numbers[3]. It is also assumed that the initial radius in each toroidal direction is the same, and comparable to the self-dual radius $\alpha'^{1/2}$. Initially, all directions are expanding isotropically with $R > \alpha'^{1/2}$ (the extension to anisotropic initial conditions has recently been considered in [161]).

As shown in this letter, as a consequence of T-duality symmetry, brane gas cosmology provides a background evolution without cosmological singularities. The scenario also provides a possible dynamical explanation for why only three spatial dimensions can become large. Winding modes (and thus T-duality) play a crucial role in the argument. Let us first focus on the winding modes of fundamental strings [29].

The winding and anti-winding modes ω and $\bar{\omega}$ of the strings are initially in thermal equilibrium with the other states in the string gas. Thermal equilibrium

[1] In this sense, brane gas cosmology is completely different in ideology than brane world scenarios in which it is assumed (in general without any dynamical explanation) that we live on a specific brane embedded in a warped bulk space-time. From the point of view of heterotic M theory [77], our considerations should be viewed as applying to the 10-dimensional orbifold space-time on which we live.

[2] For consistency, critical superstring theories need a 10 dimensional target space-time which is in apparent contradiction with the observed four. Usually it is assumed that six dimensions are compactified from the outset due to some unknown physics. However, following the usual approach in cosmology it seems more natural that initially all nine spatial dimensions were compact and small, and that three of them have grown large by a dynamical *decompactification* process. A corresponding scenario was originally proposed in [29].

[3] Recently, the scenario of [29, 5] was generalized [57, 58] to spatial backgrounds such as Calabi-Yau manifolds which admit 2-cycles but no 1-cycles.

is maintained by the process

$$\omega + \bar{\omega} \rightleftarrows \text{loops, radiation.} \tag{11.2}$$

When strings cross each other, they can intercommute such that a winding and an anti-winding mode annihilate, producing fundamental string loops or radiation without winding number. This process is analogous to infinite cosmic strings intersecting and producing cosmic string loops and radiation during their interaction (see *e.g.* [158, 28] for reviews of cosmic string dynamics).

As the spatial sections continue to expand, matter degrees of freedom will gradually fall out of equilibrium. In the context of string gas cosmology with $R > \alpha'^{1/2}$, the winding strings are the heaviest objects and will hence fall out of equilibrium first. Since the energy of a winding mode is proportional to R, Newtonian intuition would imply that the presence of winding modes would prevent further expansion. This is contrary to what would be obtained by using the Einstein equations. However, the equations of dilaton gravity yield a similar result to what is obtained from Newtonian intuition [153]: the presence of winding modes (with negative pressure) acts as a confining potential for the scale factor.

As long as winding modes are in thermal equilibrium, the total energy can be minimized by transferring it to momentum or oscillatory modes (of the fundamental string). Thereby, the number of winding modes decreases, and the expansion can go on. However, if the winding modes fall out of equilibrium, such that there is a large number of them left, the expansion is slowed down and eventually stopped. If we now try to make d of the original 9 spatial dimensions much larger than the string scale, then an obstruction is encountered if $d > 3$: in this case the probability for crossing and therefore for equilibrating according to the process (11.2) is zero. On the other hand, in a three-dimensional subspace of the nine-torus, two strings will generically meet. Therefore, the winding modes can annihilate, thermal equilibrium can be maintained, and, since the decay modes of the winding strings have positive pressure, the expansion can go on[4]. As a result, there is no topological obstruction to three dimensions of the torus growing large while the other six are staying small of size $R \sim \alpha'^{1/2}$. Large compact dimensions are not in contradiction with observations if their radius is bigger than the Hubble radius today.

It is not hard to include p-branes into the above scenario [5]. Now the initial state is a hot, dense gas of all branes allowed in a particular theory. In particular, brane winding modes are excited, in addition to modes corresponding to fluctuations of the brane. Since the winding modes play the most important role in the dynamical decompactification mechanism of [29, 5], we will focus our attention on these modes. The analogous classical counting argument as given above for strings yields the result that p-brane winding modes can interact in at most $2p+1$

[4]Since quantum mechanically, the thickness of the strings is given by the string length [89], it is important for the brane gas scenario that the initial size of the spatial sections was string scale. Otherwise, it would always be the total dimensionality of space which would be relevant in the classical counting argument of [29], and there could be no expansion in any direction.

spatial dimensions. Since for weak string coupling and for spatial sizes larger than the self-dual radius the mass of a p-brane (with a fixed winding number in all of its p spatial dimensions) increases as p increases, p-branes will fall out of equilibrium earlier the larger p is. Thus, for instance in a scenario with 2-branes, these will fall out of equilibrium before the fundamental strings and allow five spatial dimensions to start to grow [5]. Within these five spatial dimensions, the fundamental string winding modes will then allow only a three-dimensional subspace to become macroscopic. Thus there is no topological obstruction to the dynamical generation of a hierarchy of internal dimensions.

11.3 Energy, momentum, and pressure of p-branes

This section is devoted to the derivation of physical quantities describing the brane gas which determine the cosmological evolution of the background space-time.

Starting from the Nambu–Goto action, we obtain expressions for the energy and the momentum of a p-brane in $D = d + 1$ dimensional space-time. We show that there is no momentum flowing along the p tangential directions. From the energy–momentum tensor one can also define a pressure, and hence an equation of state, for the whole brane gas. Even though some of the results in this section are already known, we find it useful to give a self-consistent overview.

Let $\sigma = (\sigma^0, \sigma^i), i = 1, \cdots, p$, denote intrinsic coordinates on the world-sheet of a p-brane. Its position (or embedding) in D-dimensional space time is described by $x^M = X^M(\sigma)$, where $M = 0, \cdots, d$, and the induced metric is $g_{\mu\nu} = \eta_{MN} X^M_{,\mu} X^N_{,\nu}$, where $\mu, \nu = 0, \cdots, p$.

In the string frame, the Nambu–Goto action is

$$S_p = -T_p \int \mathrm{d}^{p+1}\sigma \mathrm{e}^{-\Phi} \sqrt{-g}\,, \tag{11.3}$$

where T_p denotes the tension of a p-brane, $g = \det(g_{\mu\nu})$, and Φ is the dilaton of the compactified theory. For our adiabatic considerations, we assume that it is constant (taking its asymptotic value), and absorb it into a physical tension

$$\tau_p = \mathrm{e}^{-\Phi} T_p = \frac{T_p}{g_s} = \frac{1}{(2\pi)^p} \frac{1}{g_s \alpha'^{(p+1)/2}}\,, \tag{11.4}$$

where in the final step we have used the explicit expression for T_p (see *e.g.* [125]). Note that for any p-brane τ_p goes like $1/g_s$, and that hence, in the weak string coupling regime which we are considering, the branes are heavy. The action (11.3) can be written as an integral over D-dimensional space-time

$$S_p = \int \mathrm{d}^D x \left(-\tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma)) \sqrt{-g} \right)\,. \tag{11.5}$$

As the integration domain is a torus, both integrals are finite.

Varying the action (11.5) with respect to the background metric and comparing with the usual definition of the space-time energy-momentum tensor, one obtains[5]

$$T^{MN}(x^K) = -\tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma))\sqrt{-g}\, g^{\mu\nu} X^M_{,\mu} X^N_{,\nu}\,. \tag{11.6}$$

The Nambu–Goto action (11.3) is invariant under $p+1$ reparametrizations $\sigma \to \tilde{\sigma}(\sigma)$, and we can use this freedom to choose

$$g_{00} = -\sqrt{-g}\,, \qquad g_{0i} = 0\,. \tag{11.7}$$

Notice that for the spatial sub-matrix with indices $i, j, k = 1, \cdots, p$, $\det(g_{ij}) = -\sqrt{-g}$ and $g^{ik} g_{kj} = \delta^i_j$. By choosing the gauge (11.7), we do not specify a particular embedding. This will be convenient later when treating a brane gas, where the branes have arbitrary orientations. Furthermore, it is consistent to set $X^0 = \sigma^0$.

To calculate the energy E_p of a p-brane, one observes that in the gauge (11.7)

$$\begin{aligned} g^{\mu\nu} X^0_{,\mu} X^0_{,\nu} &= g^{00} = -\frac{1}{\sqrt{-g}} \\ \Rightarrow \qquad T^{00}(x^K) &= \tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma))\,. \end{aligned} \tag{11.8}$$

Writing $x^K = (t, x^n), n = 1, \cdots, d$, and splitting the delta-function into a product, the integral over σ^0 can be carried out. The energy density of a p-brane in d spatial dimensions is then

$$\rho_p \equiv T^{00}(t, x^n) = \tau_p \int \mathrm{d}^p\sigma \delta^{(d)}(x^n - X^n(t, \sigma^i)), \tag{11.9}$$

and its total energy is

$$E_p = \int \mathrm{d}^d x\, \rho_p = \tau_p \int \mathrm{d}^p \sigma = \tau_p \mathrm{Vol}_p\,. \tag{11.10}$$

The volume of a p-brane in its rest frame, Vol_p, is finite as the brane is wrapped around a torus. Eq. (11.10) provides a formula for the lowest mass state, $M_p = E_p$, which will be used in section 11.5. As expected intuitively, the minimal mass is equal to the tension times the volume of a brane.

To calculate the space-time momentum P^n_p of a p-brane, one first evaluates

$$\begin{aligned} g^{\mu\nu} X^0_{,\mu} X^n_{,\nu} &= -\frac{X^n_{,0}}{\sqrt{-g}} \\ \Rightarrow \qquad T^{0n}(x^K) &= \tau_p \int \mathrm{d}^{p+1}\sigma \delta^{(D)}(x^K - X^K(\sigma)) X^n_{,0}\,. \end{aligned} \tag{11.11}$$

[5]In curved backgrounds, with metric G_{MN} say, the energy-momentum tensor gets multiplied with $\sqrt{-\det(G_{MN})}$.

Proceeding similarly as before, the total momentum of a p-brane is found to be

$$P_p^n = \tau_p \int \mathrm{d}^p\sigma \dot{X}^n(t,\sigma^i)\,, \tag{11.12}$$

where the dot denotes the derivative with respect to time t. The gauge conditions (11.7) can be written as $0 = g_{0i} = \dot{X}^m X_{m,i}$, where the sum over $m = 1, \cdots, d$ is the ordinary Euclidean scalar product. This is equivalent to saying that the (spatial) velocity vector $\dot{X}$ is perpendicular onto each of the tangential vectors $X^m_{,i}$. Therefore, only the transverse momentum is observable[6]. Assuming that the brane is a pointlike classical object with respect to the transverse directions, this momentum is not quantized despite of the compactness of space. In particular, the question whether there might exist a T-duality correspondence between transverse momentum modes and winding modes does not arise. Moreover, we neglect the possibility of open strings travelling on the brane which would in fact lead to a non zero tangential momentum[7]. Hence, in what follows, we focus on the zero modes of p-branes.

Finally, the pressure $\mathcal{P}_p$ of a p-brane is given by averaging over the trace T^m_m. First, notice that

$$\begin{aligned}
& g^{\mu\nu}X^m_{,\mu}X_{m,\nu} \\
= \quad & g^{00}X^m_{,0}X_{m,0} + g^{11}g_{11} + \cdots + g^{pp}g_{pp} \\
& +2g^{12}g_{21} + \cdots + 2g^{1p}g_{p1} \\
& +2g^{23}g_{32} + \cdots + 2g^{2p}g_{p2} + \cdots \\
& +2g^{p-1,p}g_{p,p-1} \\
= \quad & -\frac{1}{\sqrt{-g}}X^m_{,0}X_{m,0} + p\,.
\end{aligned} \tag{11.13}$$

In the first step we have used the fact that the products of the embedding functions can be expressed in terms of the induced metric, for instance $g_{11} = X^m_{,1}X_{m,1}$, and in the second step that $g^{ik}g_{kj} = \delta^i_j$. Inserting this into Eq. (11.6), eliminating the remaining $\sqrt{-g}$ by using $-\sqrt{-g} = g_{00} = -1 + X^m_{,0}X_{m,0}$, and integrating out the σ^0 dependence, one finds

$$T^m_m(t,x^n) = \tau_p \int \mathrm{d}^p\sigma \delta^{(d)}(x^n - X^n(t,\sigma^i))[(p+1)\dot{X}^m\dot{X}_m - p]\,. \tag{11.14}$$

The quantity $\dot{X}^m\dot{X}_m$ is the squared velocity of a point on a brane parameterized by (t,σ^i). We define the mean squared velocity of the branes in the gas by averaging over all σ^i, *i.e.* $v^2(t) \equiv \langle \dot{X}^m\dot{X}_m\rangle$. In the averaged trace, $\langle T^m_m\rangle$, the velocity term can be taken out of the integral. Comparing with Eq. (11.9), one

[6]Note the analogy with topological defects in field theory, where also only the transverse momentum of the defects - here taken to be straight - is observable.

[7]A relation between brane winding modes and open string momentum modes, quantized as n_m/R_m, was shown in [140].

obtains the equation of state of a p-brane gas

$$\mathcal{P}_p \equiv \frac{1}{d}\langle T^m_m \rangle = \left[\frac{p+1}{d}v^2 - \frac{p}{d}\right]\rho_p \, . \tag{11.15}$$

In the relativistic limit, $v \to 1$, the branes behave like ordinary relativistic particles: $\mathcal{P}_p = \frac{1}{d}\rho_p$, whereas in the non relativistic limit, $v \to 0$, $\mathcal{P}_p = -\frac{p}{d}\rho_p$. For domain walls this result was obtained in [165].

The pressure $\mathcal{P}_p$ and the energy density ρ_p are the source terms in the Einstein equations for the brane gas [5, 153].

11.4 Winding states and T-duality

We briefly review some of the properties of T-duality that are needed subsequently.

Consider a nine-torus T^9 with radii $(R_1, \cdots, R_9)$. Under a T-duality transformation in n-direction

$$R_n \to R_n' = \frac{\alpha'}{R_n}\,, \tag{11.16}$$

and all other radii stay invariant. T-duality also acts on the dilaton (which is constant in our case), and hence on the string coupling constant, as

$$g_s \to g_s' = \frac{\alpha'^{1/2}}{R_n} g_s \, . \tag{11.17}$$

Note, however, that the fundamental string length $l_s \equiv \alpha'^{1/2}$ is an invariant. The transformation law (11.17) follows from the requirement that the gravitational constant in the effective theory remains invariant under T-duality. In general, T-duality changes also the background geometry. However, a Minkowski background (as we are using here) is invariant.

For a p-brane on T^9, a particular winding state is described by a vector $\omega = (\omega_1, \cdots, \omega_9)$. There are $9!/[p!(9-p)!]$ such vectors corresponding to all possible winding configurations. For illustration take a 2-brane on a three-torus: it can wrap around the (12), (13), (23) directions, and hence there are $3!/2! = 3$ vectors $\omega = (\omega_1, \omega_2, 0), \omega = (\omega_1, 0, \omega_3), \omega = (0, \omega_2, \omega_3)$.

Whereas T-duality preserves the nature of a fundamental string, it turns a p-brane into a different object. To see this consider a brane with p single windings $\omega = (1, \cdots, 1, 0, \cdots, 0)$ which represents a p-dimensional hypersurface on which open strings end. Along the brane the open string ends are subject to Neumann boundary conditions. These become Dirichlet boundary conditions on the T-dual coordinate R_n' (if n denotes a tangential direction), *i.e.* for each string endpoint R_n' is fixed. Thus a T-duality in a tangential direction turns a p-brane into a (p-1)-brane. Similarly, a T-duality in an orthogonal direction turns it into a (p+1)-brane (see *e.g.* [125] for more details).

Next, consider a T-duality transformation in a direction in which the p-brane has multiple windings $\omega_n > 1$. One obtains a number ω_n of (p-1)-branes which are equally spaced along this direction. As an example take a 1-brane with winding $\omega_1 = 2$ on a circle with radius R_1. This configuration is equivalent to a 1-brane with *single* winding on a circle with radius $2R_1$. T-dualizing in 1-direction gives a single 0-brane on a circle of radius $\alpha'/2R_1$ which is equivalent to two 0-branes on a circle of radius α'/R_1 (see *e.g.* [75]). Since applying a T-duality transformation twice in the same direction yields the original state (up to a sign in the Ramond-Ramond fields), also the inverse is true: a number ω_n of (p-1)-branes correspond to a single p-brane with winding ω_n.

So far we have discussed T-duality transformations in a single direction. For applications to isotropic brane gas cosmology we need to consider T-dualizing in all nine spatial directions. Given a gas of branes, $\mathcal{B}$, on a nine-torus with radii $(R_1, \cdots, R_9)$ consisting of a large number of branes of all types admitted by a particular string theory, we want to find the corresponding gas $\mathcal{B}^*$ on the dual torus T^* with radii $(R'_1, \cdots, R'_9)$. To that end one performs a T-duality transformation in each of the nine spatial directions. From what we have discussed so far, it is now easy to see that a p-brane in a winding state $\omega = (\omega_1, \cdots, \omega_p, 0, \cdots, 0)$ is mapped into a number $\omega_1 \cdots \omega_p$ of (9-p)-branes, each of which is in a state $\omega^* = (0, \cdots, 0, 1, \ldots, 1)$. The (9-p)-brane wraps in the (9-p) directions orthogonal to the original p-brane. It is clear that the above considerations hold for any winding configuration.

After these preparatory steps, we now turn to the main part of this letter.

11.5 Mass spectra and T-duality

11.5.1 Masses of p-branes with single winding

In this section we show that each mass state in a brane gas $\mathcal{B}$ has a corresponding state with equal mass in the brane gas $\mathcal{B}^*$. Based on type IIA superstring theory we take $\mathcal{B}$ to consist of 0, 2, 4, 6 and 8-branes. Then, by the discussion in the preceding section, the brane gas $\mathcal{B}^*$ contains 9, 7, 5, 3, 1-branes which are the states of type IIB as we have carried out an odd number (nine) of T-duality transformations. Notice that this follows from the T-duality symmetry for fundamental strings, not from T-duality arguments applied to the above brane gases which we actually want to show. Our demonstration is done by carrying out explicitly nine T-duality transformations on a mass state in $\mathcal{B}$, and showing that there is a corresponding and equal mass state in $\mathcal{B}^*$. In this sense the two brane gases are T-dual.

Suppose that the branes in $\mathcal{B}$ are wrapped around some of the cycles of a nine-torus with radii $(R_1, \cdots, R_9)$. Then, the volume Vol_p of a p-brane in Eq. (11.10) is simply the product of the p circumferences, and the minimal masses $M_p = E_p$

(in the string frame) are

$$M_0 = \frac{1}{g_s \alpha'^{1/2}}\,, \tag{11.18}$$

$$M_2 = (2\pi)^2 R_9 R_8 \tau_2 = \frac{R_9 R_8}{g_s \alpha'^{3/2}}\,, \tag{11.19}$$

$$M_4 = (2\pi)^4 R_9 R_8 R_7 R_6 \tau_4 = \frac{R_9 R_8 R_7 R_6}{g_s \alpha'^{5/2}}\,, \tag{11.20}$$

$$M_6 = (2\pi)^6 R_9 \cdots R_4 \tau_6 = \frac{R_9 \cdots R_4}{g_s \alpha'^{7/2}}\,, \tag{11.21}$$

$$M_8 = (2\pi)^8 R_9 \cdots R_2 \tau_8 = \frac{R_9 \cdots R_2}{g_s \alpha'^{9/2}}\,, \tag{11.22}$$

(see also [125]) where in the second step we have used expression (11.4) for the tension of a p-brane. For notational convenience we have fixed a particular winding configuration. The argument is generalized for arbitrary winding configurations and winding numbers at the end of this section. If, as we have assumed, $R_n > \alpha'^{1/2}$, then the heaviest object in the theory is the 8-brane.

The Nambu–Goto action (11.3) is invariant under T-duality. Hence, all formulae derived from it (energy, mass and pressure) are valid in both the original brane gas $\mathcal{B}$ and in the dual brane gas $\mathcal{B}^*$. Thus, the mass spectrum of the $\mathcal{B}^*$ brane gas is

$$M_9^* = (2\pi)^9 R_9' \cdots R_1' \tau_9^* = \frac{R_9' \cdots R_1'}{g_s^* \alpha'^{10/2}}\,, \tag{11.23}$$

$$M_7^* = (2\pi)^7 R_7' \cdots R_1' \tau_7^* = \frac{R_7' \cdots R_1'}{g_s^* \alpha'^{8/2}}\,, \tag{11.24}$$

$$M_5^* = (2\pi)^5 R_5' \cdots R_1' \tau_5^* = \frac{R_5' \cdots R_1'}{g_s^* \alpha'^{6/2}}\,, \tag{11.25}$$

$$M_3^* = (2\pi)^3 R_3' R_2' R_1' \tau_3^* = \frac{R_3' R_2' R_1'}{g_s^* \alpha'^{4/2}}\,, \tag{11.26}$$

$$M_1^* = 2\pi R_1' \tau_1^* = \frac{R_1'}{g_s^* \alpha'}\,. \tag{11.27}$$

Since $R_n^{'} < \alpha'^{1/2}$, the heaviest brane of the dual gas $\mathcal{B}^*$ is now the 1-brane. The coupling constant in $\mathcal{B}^*$ is given by

$$g_s^* = \frac{\alpha'^{9/2}}{R_9 \cdots R_1} g_s\,. \tag{11.28}$$

Note that if the radii $(R_1, \cdots, R_9)$ of the initial nine-torus are bigger than the self-dual radius $\alpha'^{1/2}$, then $g_s^* < g_s$, and thus the assumption of a small string coupling constant is safe.

Given the two mass spectra, one can easily verify that each state in the brane gas $\mathcal{B}$ has a corresponding state with equal mass in the dual brane gas $\mathcal{B}^*$:

$$M_{9-p}^* = M_p\,. \tag{11.29}$$

This establishes explicitly that the T-duality of the string gas used in [29] extends to the brane gas cosmology of [5].

As an explicit example, consider a 2-brane wrapped around the 8 and 9 directions. Its mass is (11.19)

$$M_2 = \frac{R_8 R_9}{g_s \alpha'^{3/2}} \,. \tag{11.30}$$

If we replace the string coupling constant g_s by the dual string coupling constant g_s^* via (11.28), and the radii R_8 and R_9 by the dual radii $R_8^{'}$ and $R_9^{'}$ via (11.16), one obtains

$$M_2 = \frac{R_1^{'} \cdots R_7^{'}}{g_s^* \alpha'^{8/2}} = M_7^* \,. \tag{11.31}$$

In the above example, we have specified a particular winding configuration for simplicity, but clearly the argument holds as well in the general case, where a p-brane wraps around some directions $n_1 \cdots n_p$:

$$M_p = (2\pi)^p R_{n_1} \cdots R_{n_p} \tau_p = \frac{R_{n_1} \cdots R_{n_p}}{g_s \alpha'^{(p+1)/2}} \,. \tag{11.32}$$

Via the same steps as in the above example, it follows that

$$\begin{aligned} M^*_{9-p} &= (2\pi)^{9-p} R'_{m_1} \cdots R'_{m_{9-p}} \tau^*_{9-p} = \frac{R'_{m_1} \cdots R'_{m_{9-p}}}{g_s^* \alpha'^{(10-p)/2}} \\ &= M_p \,, \end{aligned} \tag{11.33}$$

where $\{m_1, \cdots, m_{9-p}\} \neq \{n_1, \cdots, n_p\}$.

11.5.2 Multiple windings

Consider now a p-brane with multiple windings $\omega = (\omega_1, \cdots, \omega_p, 0, \cdots, 0)$. Its mass is

$$M_p(\omega) = \omega_1 \cdots \omega_p M_p \,. \tag{11.34}$$

In the $\mathcal{B}^*$ brane gas this corresponds to a number $\omega_1 \cdots \omega_p$ of (9-p)-branes each with winding $\omega^* = (0, \cdots, 0, 1, \cdots, 1)$ and mass M^*_{9-p}. Since $M^*_{9-p} = M_p$, the total mass of this 'multi-brane' state is equal to the mass of the original brane, namely

$$(\omega_1 \cdots \omega_p) M^*_{9-p} = M_p(\omega) \,, \tag{11.35}$$

which establishes the correspondence of $\mathcal{B}$ and $\mathcal{B}^*$ in the case of multiple windings.

One should also add fundamental strings to the brane gas $\mathcal{B}$. Since their mass squared is

$$M^2 = \left(\frac{n_n}{R_n}\right)^2 + \left(\frac{\omega_n R_n}{\alpha'}\right)^2 \,, \tag{11.36}$$

is its clear that every fundamental string state in $\mathcal{B}$ has a corresponding state in $\mathcal{B}^*$ when $n_n \leftrightarrow \omega_n$.

11.6 Cosmological implications and discussion

We have demonstrated explicitly how T-duality acts on a brane gas in a toroidal cosmological background, and have in particular shown that the mass spectrum of the theory is invariant under T-duality. Thus, the arguments of [29] which led to the conclusion that cosmological singularities can be avoided in string cosmology extend to brane gas cosmology.

Whereas T-duality does not change the nature of fundamental strings, but simply interchanges winding and momentum numbers, it changes the nature of branes: after T-dualizing in all d spatial dimensions, a p-brane becomes a (d-p)-brane which, however, was shown to have the same mass as the 'original' brane.

In [5] it was shown that the dynamical decompactification mechanism proposed in [29] remains valid if, in addition to fundamental strings, the degrees of freedom of type IIA superstring theory are enclosed. We briefly comment on the decompactification mechanism in the presence of a type IIB brane gas on a nine-torus. As before, we assume a hot, dense initial state where, in particular, the brane winding modes are excited and in thermal equilibrium with the other degrees of freedom. All directions of the torus are roughly of string scale size, l_s, and start to expand isotropically. For the 9,7, and 5-brane winding modes there is no dimensional obstruction to continuously meet and to remain in equilibrium, thereby transferring their energy to less costly momentum or oscillatory modes: these degrees of freedom do not constrain the number of expanding dimensions. However, the 3-branes allow only seven dimensions to grow further, and out of these, three dimensions can become large when 1-brane and string winding modes have disappeared. As far as the 'intercommutation' and equilibration process is concerned, the 1-branes and the strings play the same role, but since the winding modes of the former are heavier ($\frac{R}{g_s \alpha'} \gg \frac{R}{\alpha'}$ at weak coupling), they disappear earlier.

We have focused our attention on how T-duality acts on brane winding modes. However, since in a hot and dense initial state we expect all degrees of freedom of a brane to be excited, we should also include transverse fluctuations (oscillatory modes) in our considerations concerning T-duality. To our knowledge, the quantization of such modes is, however, not yet understood, and we leave this point for future studies.

Another interesting issue is to investigate how the present picture of brane gas cosmology gets modified when gauge fields on the branes are included. These correspond to $U(N)$ Chan-Paton factors at the open string ends. In this case, a T-duality in a transverse direction yields a number N of parallel (p-1)-branes at different positions [124].

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11.7 Appendix

In this appendix we would like to give some further arguments for why we believe brane gas cosmology to be non singular. First, let us recall the results of [29]. In the case of a string gas, where the strings are freely propagating on a 9-torus, it was shown that the cosmological evolution is free of singularities. The background space-time has to be compact, otherwise the thermodynamical description of strings is not sound, in particular the specific heat becomes negative at large energies. An important assumption in the derivation was that the evolution of the universe is adiabatic, *i.e.* the entropy of the string gas is constant. Making use of this assumption, one can find the temperature as a function of the scale factor, $T(R)$, without referring to the dynamics of gravity or Einstein's equations. Using a microcanonical approach, it was shown that there exists a maximum temperature, called Hagedorn temperature T_H, and hence there is no temperature singularity in string gas cosmology. The curve $T(R)$ is invariant under a T-duality transformation, $T(R) = T(\alpha'/R)$. Another crucial assumption is that the string coupling constant, g_s, is small enough such that the thermodynamical computations for free strings are applicable, and that the back-reaction of the string condensate on the background geometry can be neglected. In lack of knowledge about brane thermodynamics we simply postulate that the statements above extend to brane gases.

We conclude by making a comment on our work in the light of the well-known singularity theorems in general relativity. These theorems make assumptions about the geometry of space-time such as $R_{\mu\nu}v^\mu v^\nu \geq 0$ for all timelike vectors v^μ (for a textbook treatment see *e.g.* [160]). By Einstein's equations this is equivalent to the strong energy condition for matter. However, we do not trust Einstein's equations in the very early universe as they receive corrections which are higher order in α' as well as g_s, and also they lack invariance under T-duality transformations $R \rightarrow \alpha'/R$. Therefore we cannot invoke the energy momentum tensor given in Eqs. (11.9),(11.11), (11.14) to decide whether the universe described by our model is geodesically complete or not.

Conclusions

In this thesis, we have investigated different topics at the interface between string theory and cosmology. In the first part, we identified our universe with a 3-brane embedded in a 5-dimensional bulk. Starting with a rather simple approach to perturbations on moving branes, we have considered a dynamical generalization of the Randall-Sundrum model, and finally estimated the CMB power spectrum for anti-de Sitter brane worlds. In all of these cases, dynamical instabilities were encountered, which suggests that existing attempts to realize cosmology on branes are at least questionable.

In the second part, we considered an alternative model, in which we do not live on a particular brane. Instead, the role of strings and branes is only to 'regulate' the background dynamics. Though this approach is less well developed, it seems to us a more promising avenue to bring together string theory and cosmology. In particular, progress has been made toward the resolution of the initial singularity problem.

Common to both approaches is that a higher-dimensional view allows to discover new relations between seemingly unconnected phenomena and to understand them from a unified perspective. Below, we summarize the conclusions of each article.

Perturbations on a moving D3-brane and mirage cosmology

We have studied the evolution of perturbations on a moving 3-brane coupled to a 4-form field in an AdS_5-Schwarzschild bulk. The unperturbed dynamics lead to homogeneous and isotropic expansion on the brane, where the scale factor $a(\tau)$ is related to the position of the brane via $a \propto r$. The fluctuations we consider are due only to this motion, and can be described in terms of a scalar field ϕ. For a BPS brane[8], parameterized by a conserved energy $E = 0$, we find that **superhorizon** modes evolve as

$$\phi_k = A_k a^4 + B_k a^{-3},$$

where the constants A_k and B_k are determined by the initial conditions. Hence, if the brane is expanding, superhorizon modes grow as a^4. When the brane is contracting, superhorizon modes grow as a^{-3}. Of course, at some point our linear analysis will break down. For **subhorizon** modes we find

$$\phi_k = A_k \frac{\mathrm{e}^{ik\eta}}{a} + B_k \frac{\mathrm{e}^{-ik\eta}}{a}.$$

For an expanding brane subhorizon modes are stable whereas for a contracting brane they grow large. In particular, the brane becomes unstable as it falls into the AdS_5-Schwarzschild black-hole ($r \searrow 0, a \searrow 0$). We have also discussed the cases $E > 0$, BPS anti-branes (*i.e.* with opposite Ramond-Ramond charge) as well as non BPS branes.

[8]A BPS Dp-brane is characterized in that its charge is equal to its tension, see Eq. 5.2. In the low energy description, it corresponds to an extremal p-brane.

The perturbations ϕ gives rise to scalar perturbations in the Friedmann-Lemaître (brane) universe. For a 4-dimensional observer living on the perturbed brane, ϕ is related to the gauge invariant Bardeen potentials in the following way:

$$\Phi = -\left(\frac{\tilde{E}}{a^4} + q\right)\left(\frac{\phi}{L}\right),$$

$$\Psi = 3\Phi + 4q\left(\frac{\phi}{L}\right),$$

where $\tilde{E} \equiv E - q\frac{r_H^4}{2L^4}$, r_H is the horizon, L is the curvature radius of the AdS$_5$-Schwarzschild geometry, and q is a parameter taking the value 1 for BPS branes, and -1 for BPS anti-branes. Our approach is limited to the case in which the back-reaction of the brane onto the bulk can be neglected.

We have also derived a Friedmann equation on the 3-brane and found terms in $1/a^8, 1/a^4$ as well as a cosmological constant if the brane is non BPS. Thus, pure brane motion in an AdS$_5$-Schwarzschild background cannot mimick the effect of dust matter. It is not excluded though that in other backgrounds a term $1/a^3$ can be obtained.

Dynamical instabilities of the Randall-Sundrum model

The Randall-Sundrum model relies on a fine-tuning between the brane tension V and the (negative) cosmological constant Λ in the bulk. We have investigated whether this fine-tuning could be achieved by a dynamical mechanism. As a concrete example, we studied a scalar field on the brane, and we found that the bulk cosmological constant cannot be cancelled by the potential energy of the scalar field. More generally, this result also holds for any matter on the brane with an equation of state $w > -1$.

Furthermore, we have derived Einstein's equations for a 3-brane, which contains fluid matter and a scalar field, and which is coupled to the dilaton and to the gravitational field in a 5-dimensional bulk. They allow for a dynamical generalization of the RS model and ultimately to investigate its stability. To that end, the fine-tuning[9] is perturbed by a constant Ω,

$$V = \sqrt{-12\Lambda}(1 + \Omega).$$

We have found analytic solutions of the 5-dimensional perturbation equations, which for the scale factor on the brane yield

$$a^2(\tau) \simeq 1 + 2\mathcal{Q}\tau + 4\alpha^2\Omega\tau^2,$$

showing that a runs away from the static RS solution as τ^2 (where τ is cosmic time on the brane). In a cosmological context, the change in brane tension, Ω, may be due to a phase transition, for example.

[9] in units where $2\kappa_5^2 = 1$

More surprisingly, there exists also a mode $\mathcal{Q}$, which represents an instability, even if the fine-tuning is not perturbed at all. Within linear perturbation theory, there is no constraint on $\mathcal{Q}$, and it cannot simply be gauged away, because all our statements are gauge invariant. This mode can be understood as the velocity of the brane, thus indicating that it is not consistent to keep the brane fixed at $y = 0$ while perturbing the metric as in Eq. (7.56).

CMB anisotropies from vector perturbations in the bulk

We considered again the evolution of perturbations on a 3-brane moving in an AdS_5 bulk[10]. This time the back-reaction was taken into account by imposing Z_2 symmetry with respect to the brane position and by using Israel's junction conditions.

We have found the most general analytic solutions for vector perturbations in the bulk, which generically lead to growing vector perturbations on the brane. This differs radically from the usual behavior in 4-dimensional cosmology, where vector modes decay like a^{-2} whatever the initial conditions. We have identified vector modes on the brane, which are normalizable and exponentially growing in conformal time. Ultimately, the existence of these modes is due to Z_2 symmetry in the bulk, and they are normalizable only because the bulk is finite (see Eq. (9.51)). We comment on this issues as well as on the role of these modes in the RS model in the appendix of our article.

Regarding the vector modes, the spatial frequency of the bulk perturbations, Ω, plays the role of an effective mass, and Ω^2 might be positive or negative[11]. The latter case represents a bulk tachyonic mode, which is another way to see the instability. In addition, there exist modes which grow as a power law, but they still lead to significant effects in the CMB.

An estimate of the CMB anisotropies for the exponentially growing modes stringently constrains their primordial amplitude:

$$A_{0}(\Omega) < \mathrm{e}^{-10^{3}}, \text{ for } \Omega/a_{0} \simeq 10^{-26}\,\mathrm{mm}^{-1},$$

and

$$A_{0}(\Omega) < \mathrm{e}^{-10^{29}}, \text{ for } \Omega/a_{0} \simeq 1\,\mathrm{mm}^{-1},$$

where a_0 is the scale factor today. We have required a scale invariant spectrum around $\ell \simeq 10$, and we have set the anti-de Sitter curvature radius L to 10^{-3} mm.

Vector modes are part of gravitational waves in the bulk, which are generated for instance in bulk inflationary models. They are also likely to be produced by second-order effects in the bulk. If such processes are not forbidden by some physical mechanism, then anti-de Sitter brane worlds cannot reasonably lead to a homogeneous and isotropic expanding universe.

[10] For the calculation it turns out to be much simpler to consider a moving brane in a static background, instead of fixing the brane in a dynamical bulk.

[11] Because the Greek alphabet is finite, unfortunately Ω happened to label the perturbation of the RS fine-tuning as well as the spatial frequency of the bulk vector modes.

Since we have considered only vector perturbations here, we cannot compare our results to those found in mirage cosmology. There, the back-reaction was neglected, and so the only possible perturbations are those in the embedding, thus a single scalar. Certainly, it would be interesting to solve the scalar perturbations in the above framework, also to decide how important the inclusion of the back-reaction is.

On T-duality in brane gas cosmology

In this alternative approach, we considered a gas of strings and p-branes in a 10-dimensional background space-time, where the spatial sections have the topology of a 9-torus with radii $R_n, n = 1, \cdots, 9$. The possible values of p are either $1, 3, 5, 7, 9$ or $0, 2, 4, 6, 8$ according to the type of string theory one is considering. The principal aim of brane gas cosmology is to avoid the initial singularity. To that end, we have derived a formula for the mass of a p-brane with single or multiple windings around the cycles of the torus, as well as the equation of state for the brane gas. Consider this brane gas on a 'dual' torus, which is obtained by applying the transformation

$$R_n \to \frac{\alpha'}{R_n}$$

onto all nine spatial dimensions. This so-called T-duality transformation turns a p-brane with mass M_p into a (9−p)-brane with mass M^*_{9-p}. We have explicitly shown that

$$M^*_{9-p} = M_p \,.$$

Thus, there are the same degrees of freedom on the original and the dual torus.

For a gas of strings, invariance of the mass spectrum under a T-duality transformation has been used to show that the dynamical evolution of the torus can be described in a singularity-free way [29]. Given that this invariance holds also for branes, we argued that the initial singularity is avoided in the framework of brane gas cosmology. It remains to be shown that the temperature of the brane gas on the dual torus is the same, $T^* = T$.

We also addressed the dynamical decompactification mechanism proposed in [29], now including p-branes. In a cosmological scenario, the initial state is thought to be a hot and dense gas of strings and branes. All cycles of the torus are roughly of string scale size and start to expand isotropically. We demonstrate that the winding modes of strings and p-branes allow only a 3-dimensional subspace to become large. In this scenario, we do not live on a particular brane but in the bulk. The extra-dimensions are invisible today because they remain of string scale size.

Certainly, another attractive feature of brane gas cosmology is that the problems associated with compactification are avoided.

FIN

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